Data Analysis and Machine Learning 4 (DAML)

Week 6: Linear models for classification

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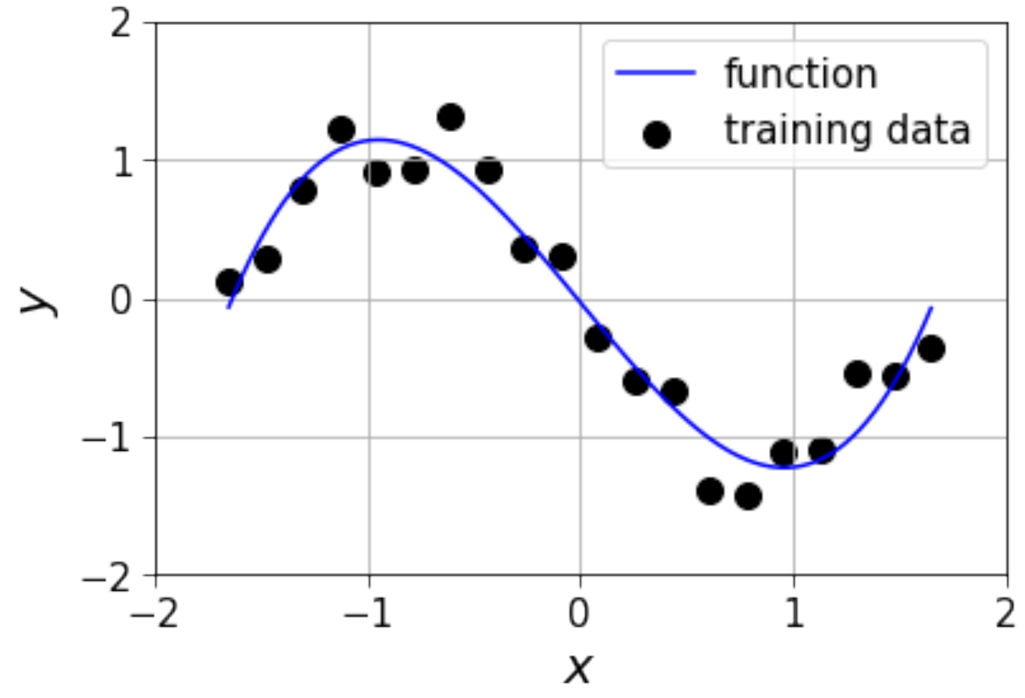
Recap

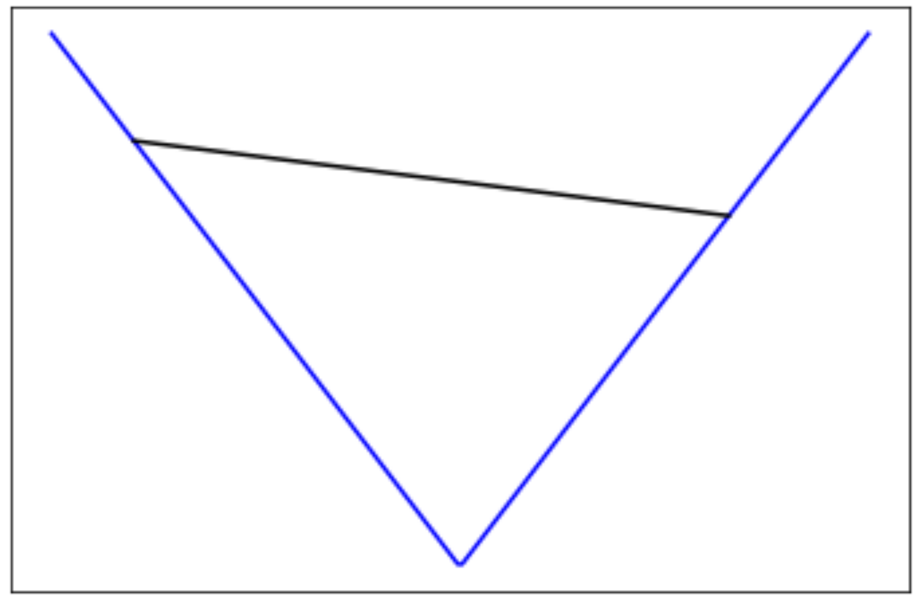
- We learned about different types of linear regression and regularisation

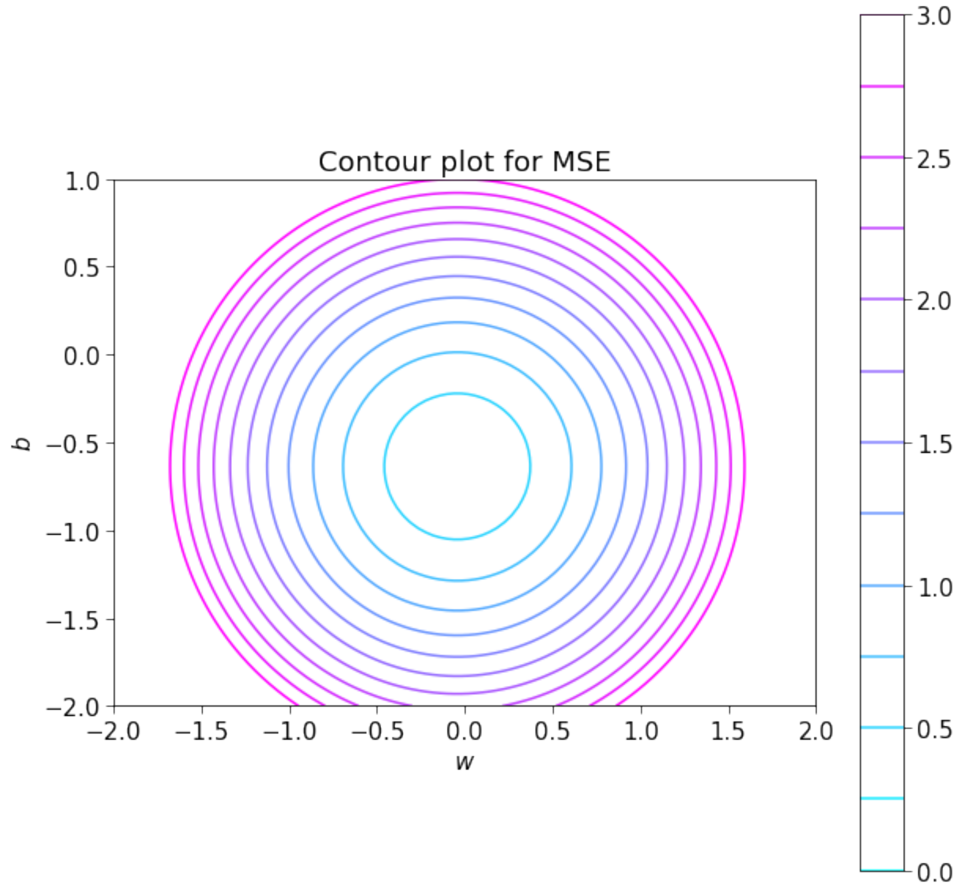


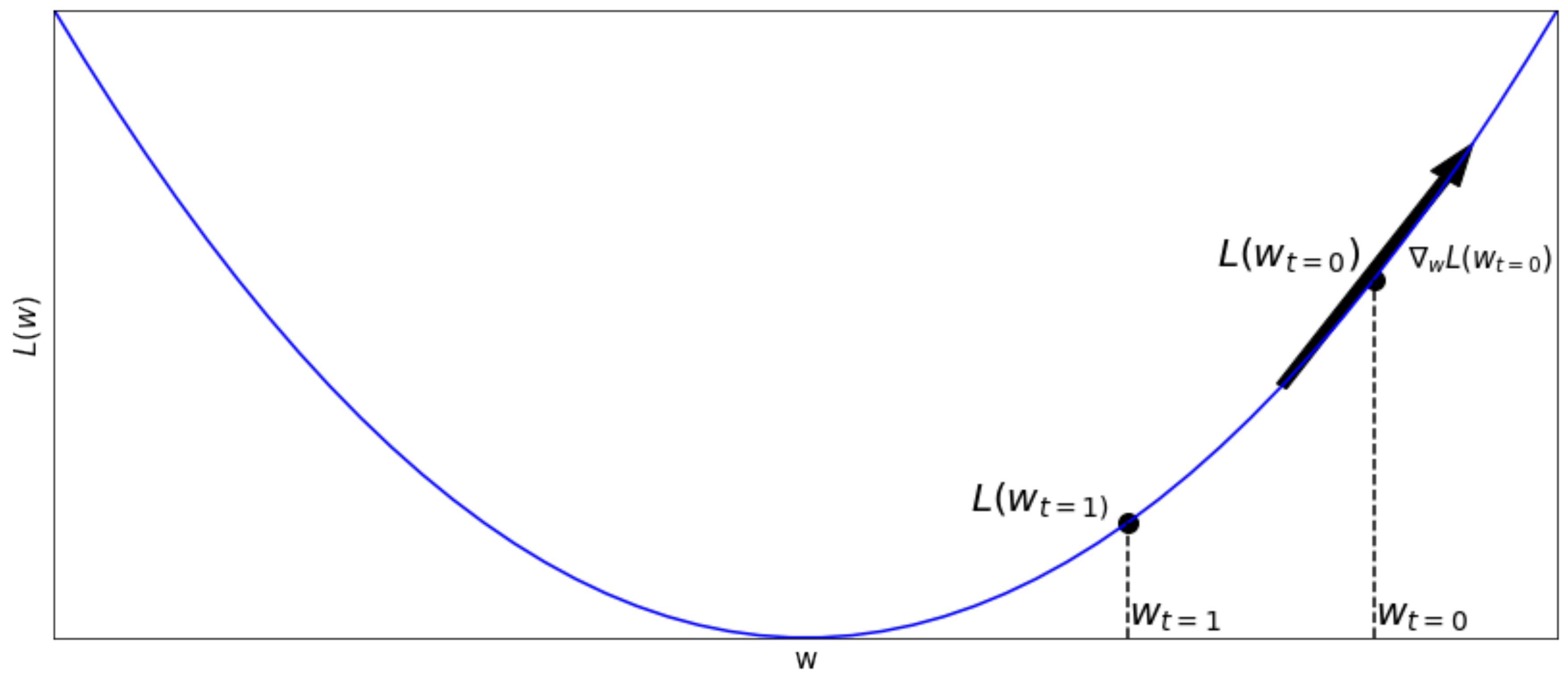
$$f(\mathbf{x}) = \hat{y} = \mathbf{w}^\top \phi(\mathbf{x})$$

$$L_{ridge}(\mathbf{w}) = \underbrace{\|\mathbf{y} - \mathbf{\Phi}\mathbf{w}\|^2}_{SE} + \underbrace{\lambda\|\mathbf{w}\|^2}_{regularisation}$$

- We looked at convex functions and gradient descent

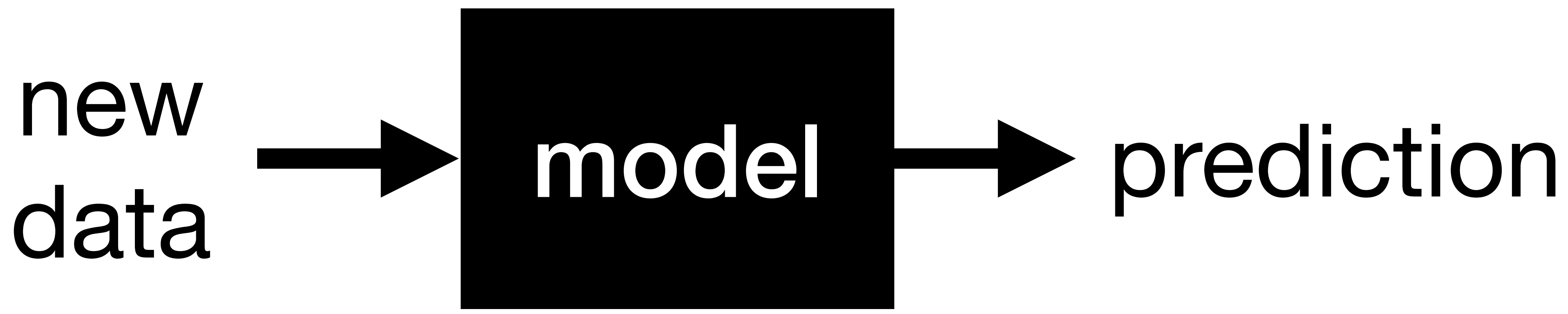






Supervised Learning

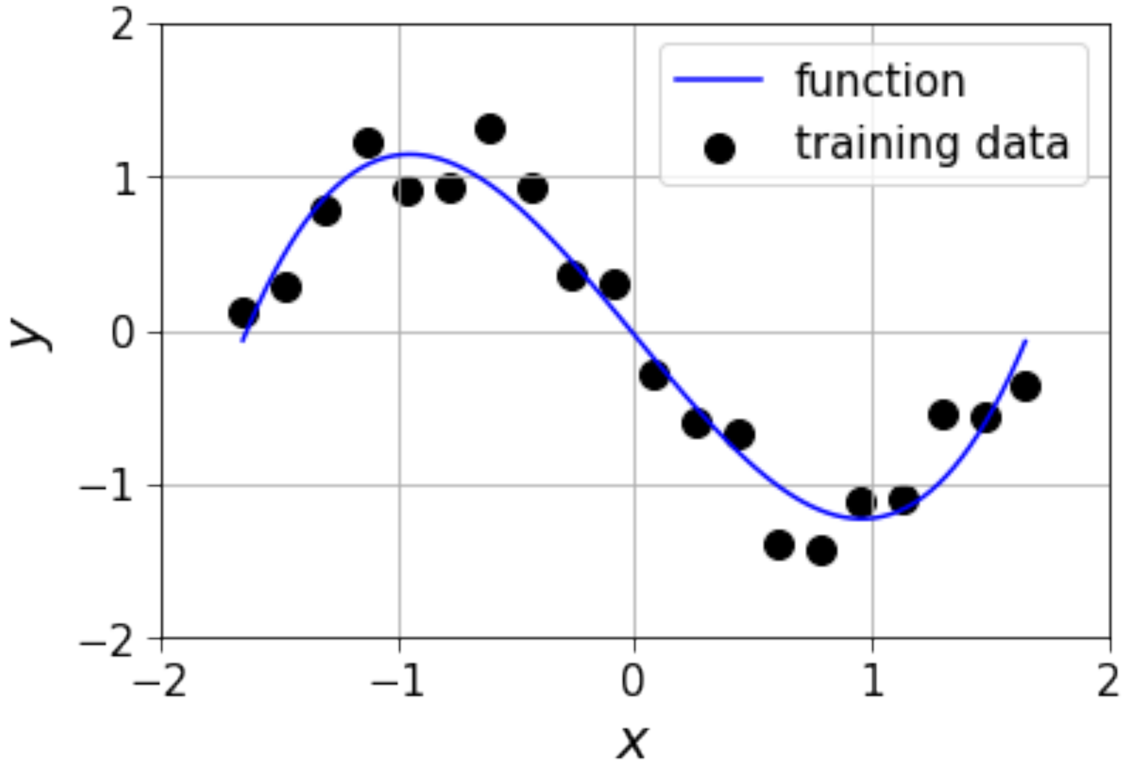
- We want a model that takes in a new data point and outputs a prediction



- For the model to be accurate it must first learn from training data
- Often, models are parameterised functions and learning = finding the best parameters
- Training data is a set of existing data points that have been **labelled**
- The label says what the prediction for that data point **should be**

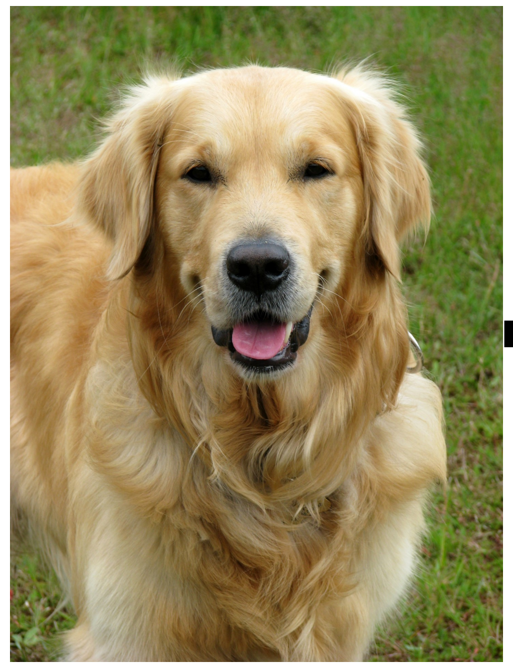
Two canonical problems in supervised learning

- **Regression**: Given input data, predict a continuous output



- **Classification**: Given input data, predict a distinct category

→ cat



→ dog

Linear models for classification

Why linear models?

- They are simple and intuitive
- They are interpretable
- They use vectors and matrices (computers love these)
- They work well in many scenarios

Slide inspired by https://sites.google.com/site/christophlampert/teaching/kernel-methods-for-object-recognition

The classification problem

- Our training set consists of N data point-target pairs $\{(\mathbf{x}^{(n)}, y^{(n)})\}_{n=1}^{N}$
- Data points $\mathbf{x} \in \mathbb{R}^D$ are column vectors, targets are class labels $y \in \mathbb{Z}^+_{<K} = \{0,1,\ldots,K-1\}$
- i.e. each data point has been labeled as belonging to 1 of K classes
- ~~Objective: We want a model that classifies our training data **correctly**~~
- **Objective:** We want a model that classifies our held-out data correctly

The most common way to quantify classification performance is **accuracy**

This is simply the fraction or % of classifications that are correct

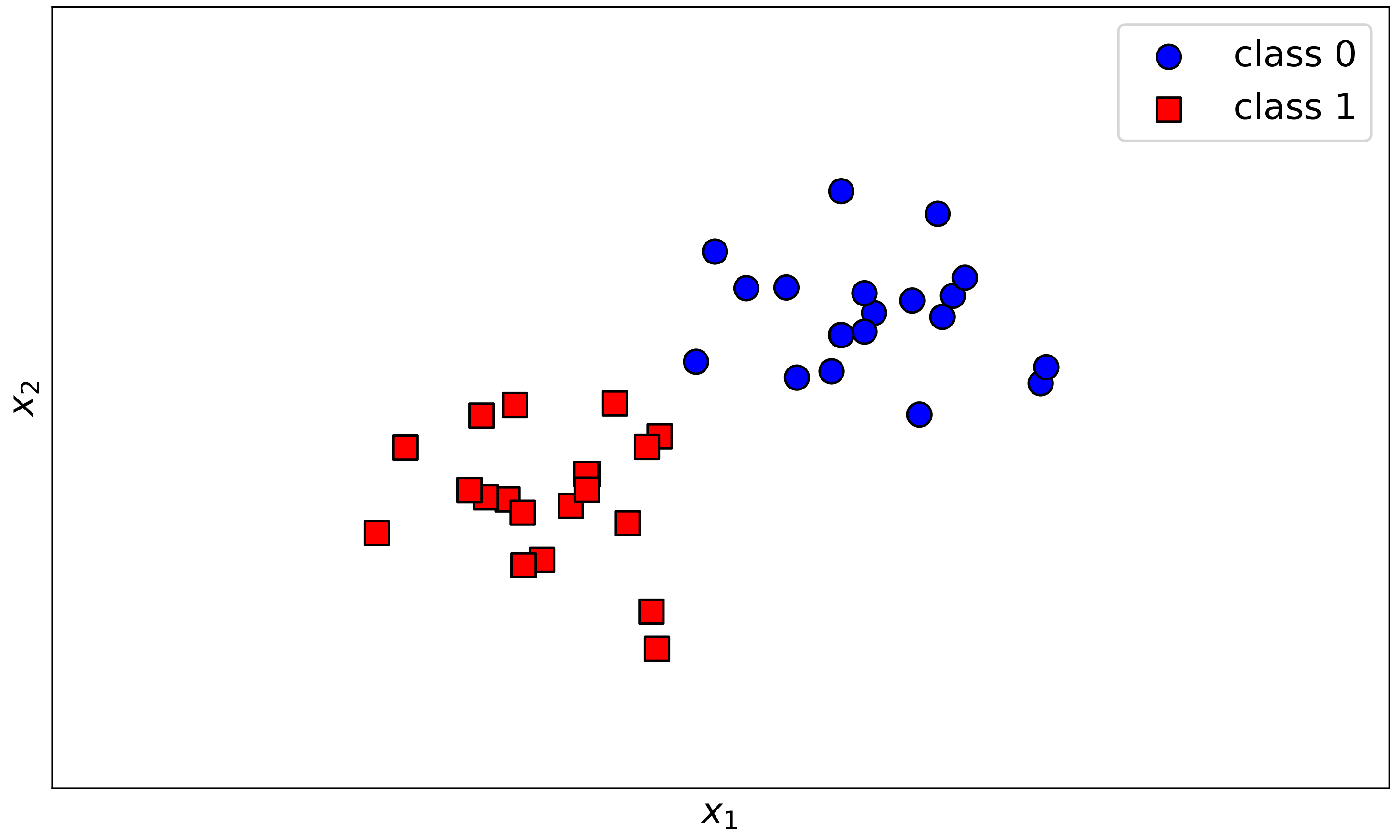
A linear classifier is a linear model + a threshold function

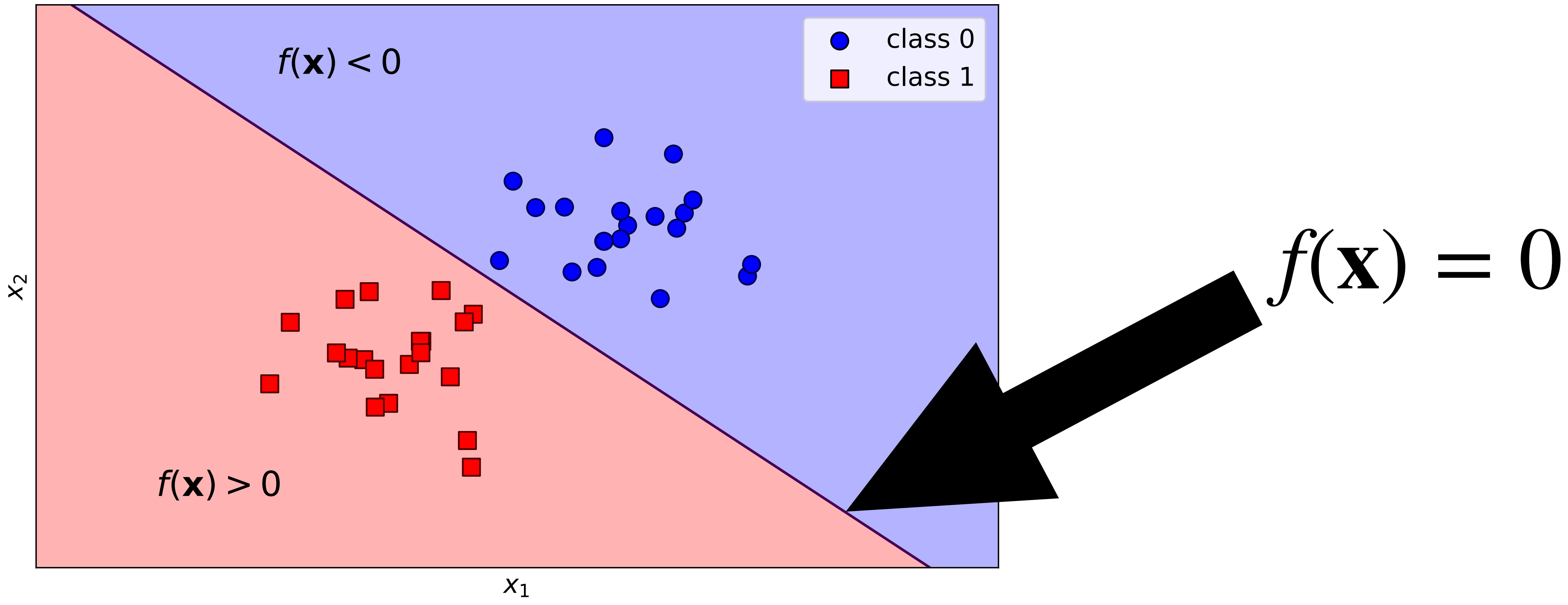
- We will use a linear model as we did for regression $f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} + b$
- For now we will consider **binary classification:** $y \in \{0,1\}$ *or* $y \in \{-1,1\}$
- For regression, we used $f(\mathbf{x}) \in \mathbb{R}$ as our target prediction but we can't do this for classification because the class labels are discrete
- Instead we will supply a **threshold function** that maps $f(\mathbf{x})$ to a discrete class prediction $\hat{y}$
- This could be $\hat{y} = \begin{cases} 1 & \text{if } f(\mathbf{x}) > 0 \\ 0 & \text{if } f(\mathbf{x}) < 0 \end{cases}$

We can call $f(\mathbf{x})$ the classifier score for $\mathbf{x}$

Linear classifier decision boundary in 2D

- Consider a training set $\{(\mathbf{x}^{(n)}, y^{(n)})\}_{n=1}^{N}$ with $\mathbf{x} \in \mathbb{R}^2$ and $y \in \{0,1\}$
- We have $f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} + b$ where $\mathbf{x} = [x_1 \quad x_2]^\top$ and $\mathbf{w} = [w_1 \quad w_2]^\top$
- Let's use the threshold function $\hat{y} = \begin{cases} 1 & \text{if } f(\mathbf{x}) > 0 \\ 0 & \text{if } f(\mathbf{x}) < 0 \end{cases}$
- The line $f(\mathbf{x}) = w_1 x_1 + w_2 x_2 + b = 0$ forms the decision boundary of the classifier

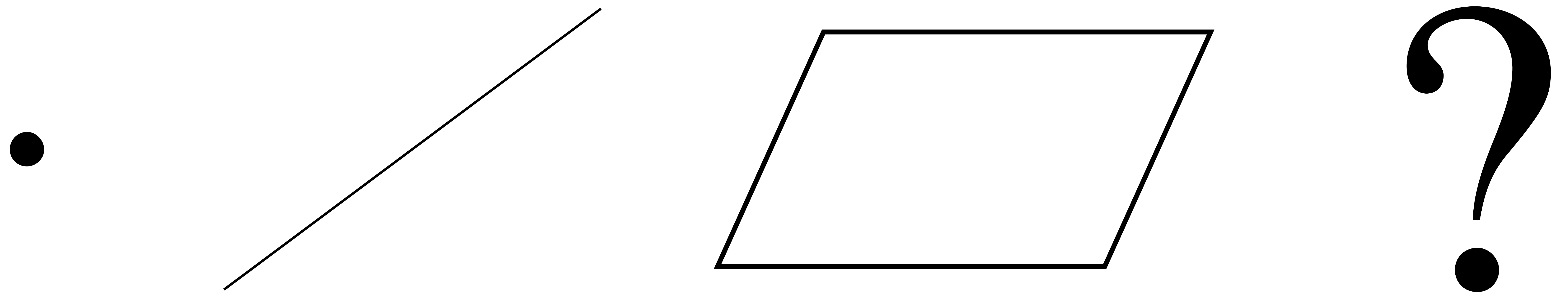




Decision boundary are hyperplanes

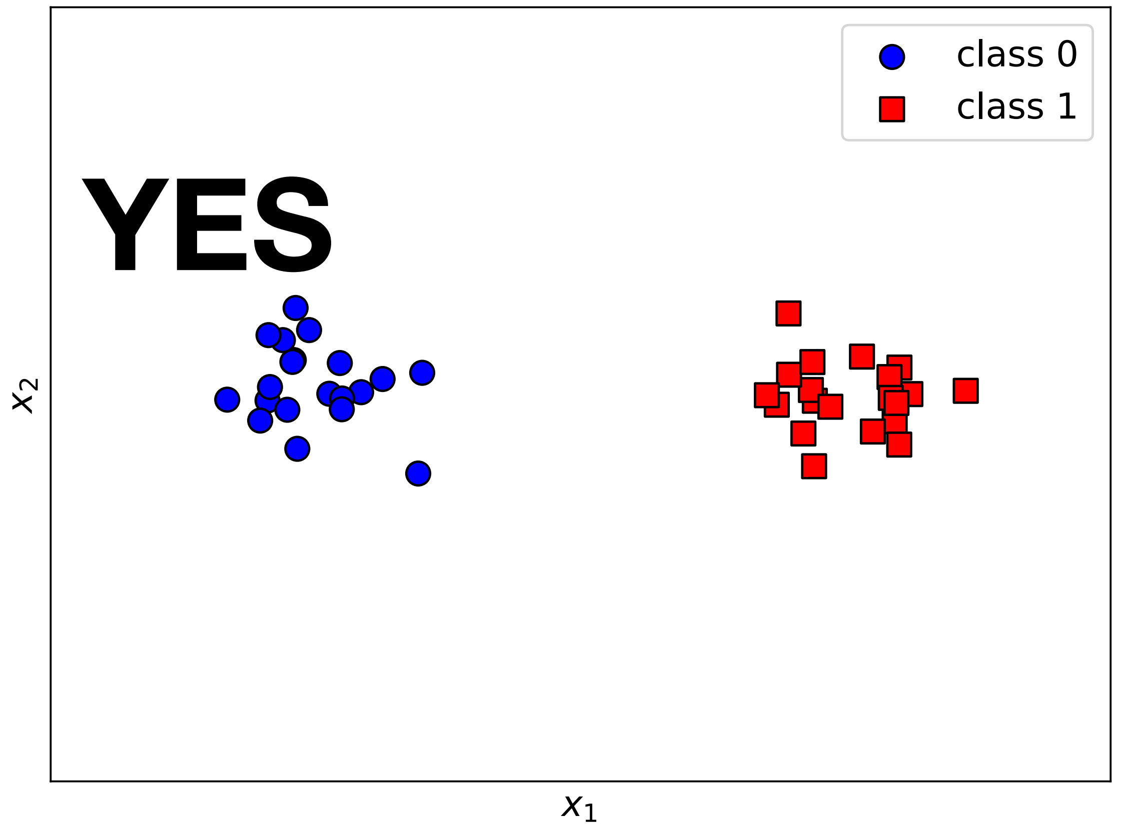
For $\mathbf{x} \in \mathbb{R}^D$ the decision boundary of a linear classifier is in $D - 1$

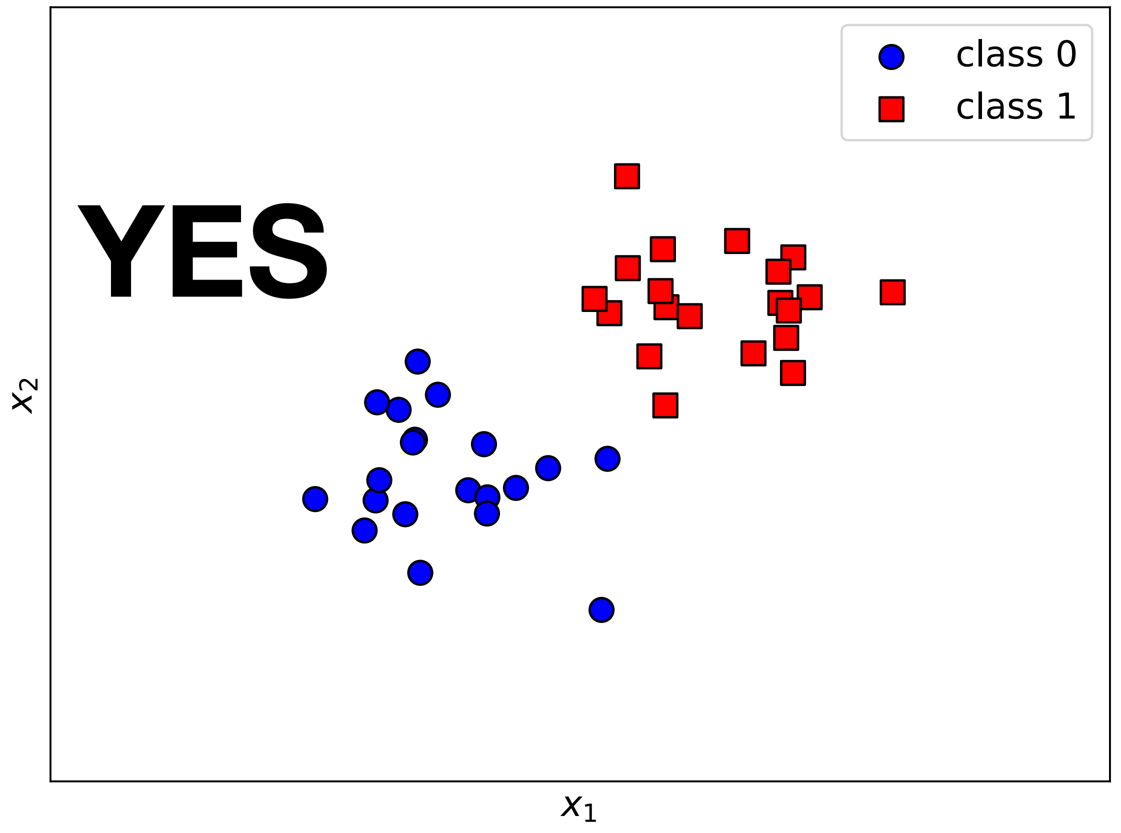
- In 1D the decision boundary is a point
- In 2D the decision boundary is a line
- In 3D the decision boundary is a plane
- In 4D and above the decision boundary is a **hyperplane** we can't visualise but all the maths still works (:

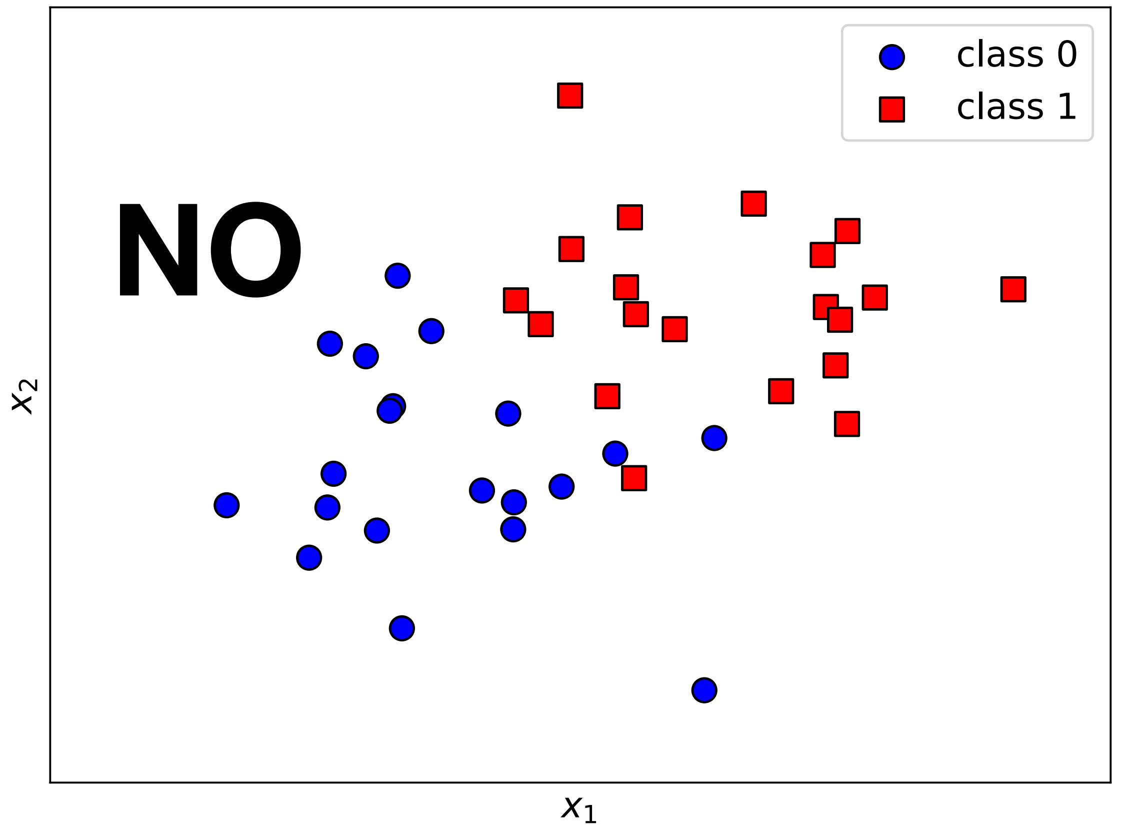


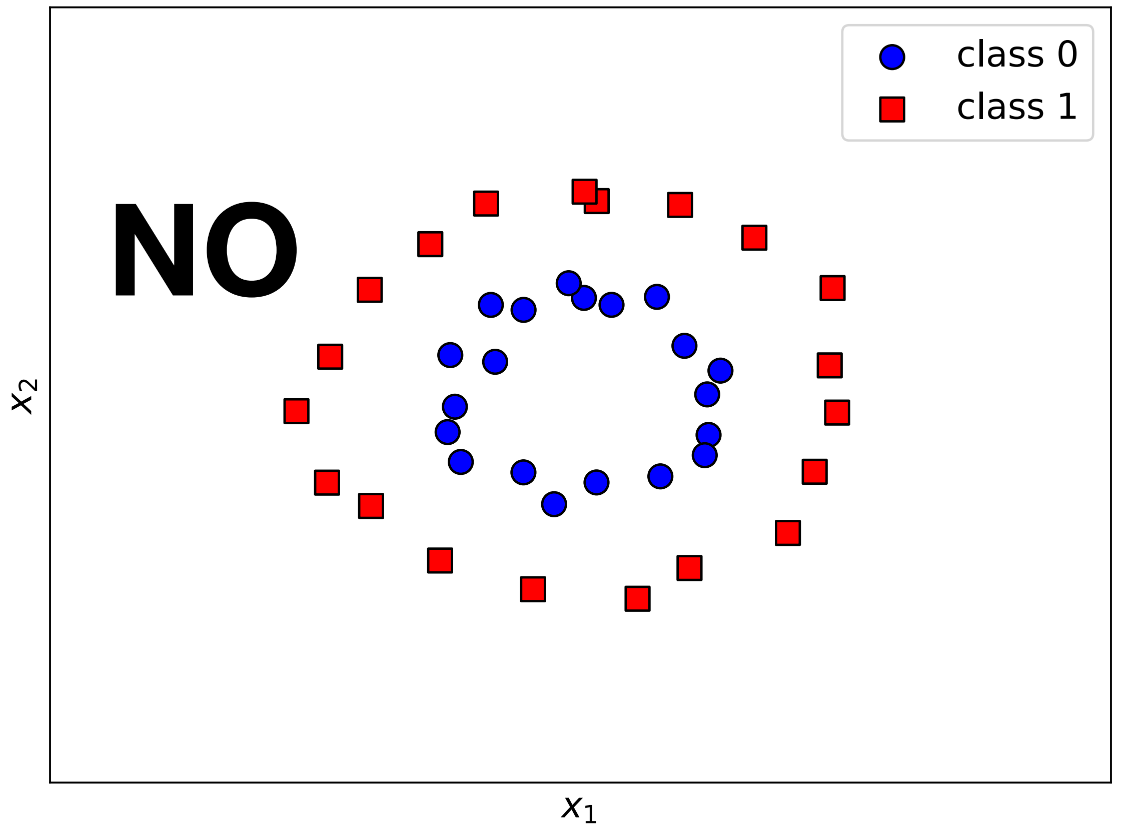
Linear separability

Our training data is linearly separable if we are able to draw a hyperplane that completely separates points from both classes



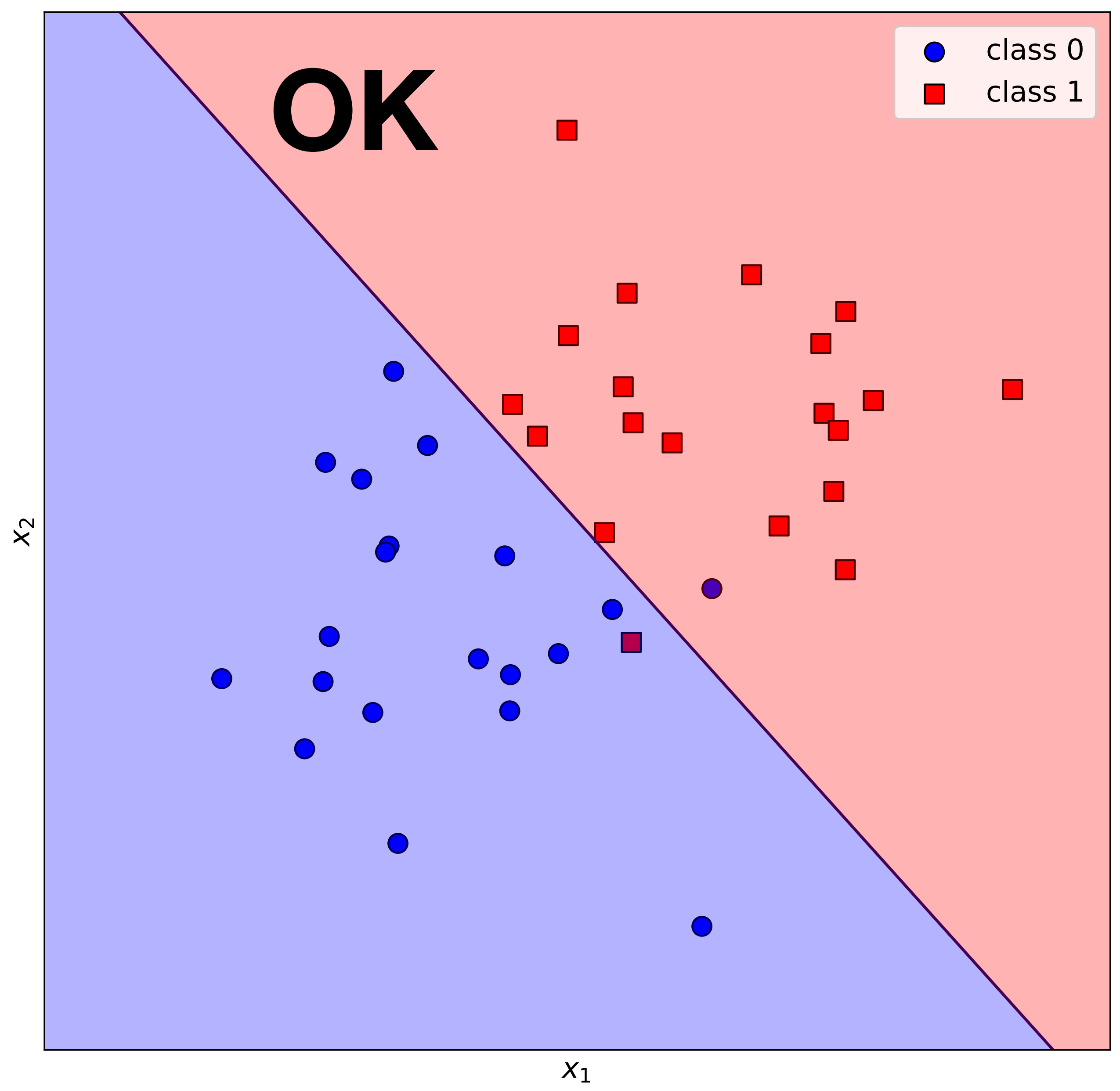


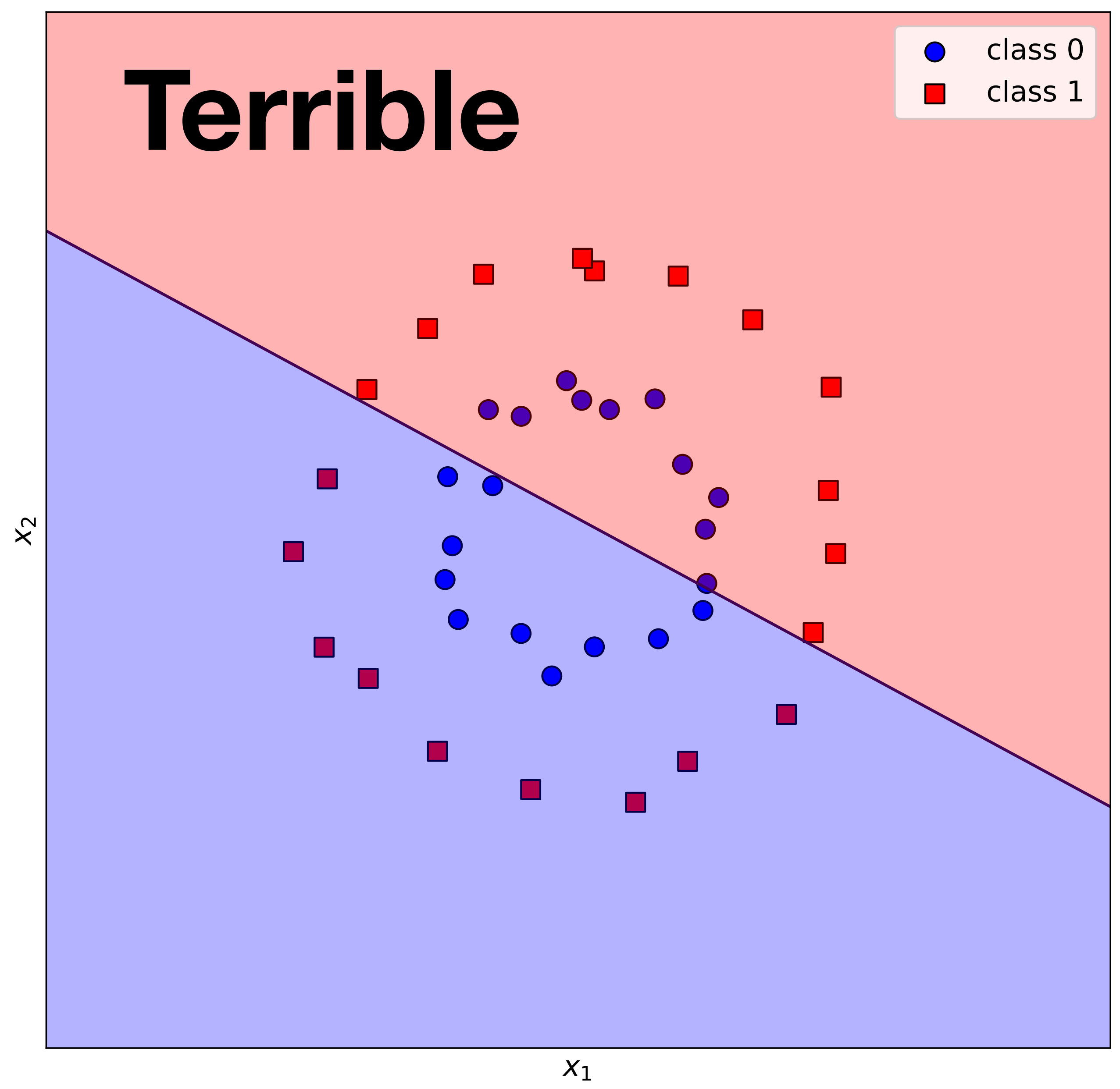




Linear separability continued

- If training data isn't linearly separable, a linear classifier can't produce a decision boundary that perfectly classifies the training data
- You can still get good solutions if a hyperplane can separate most data
- If it can't then a linear classifier won't be any good



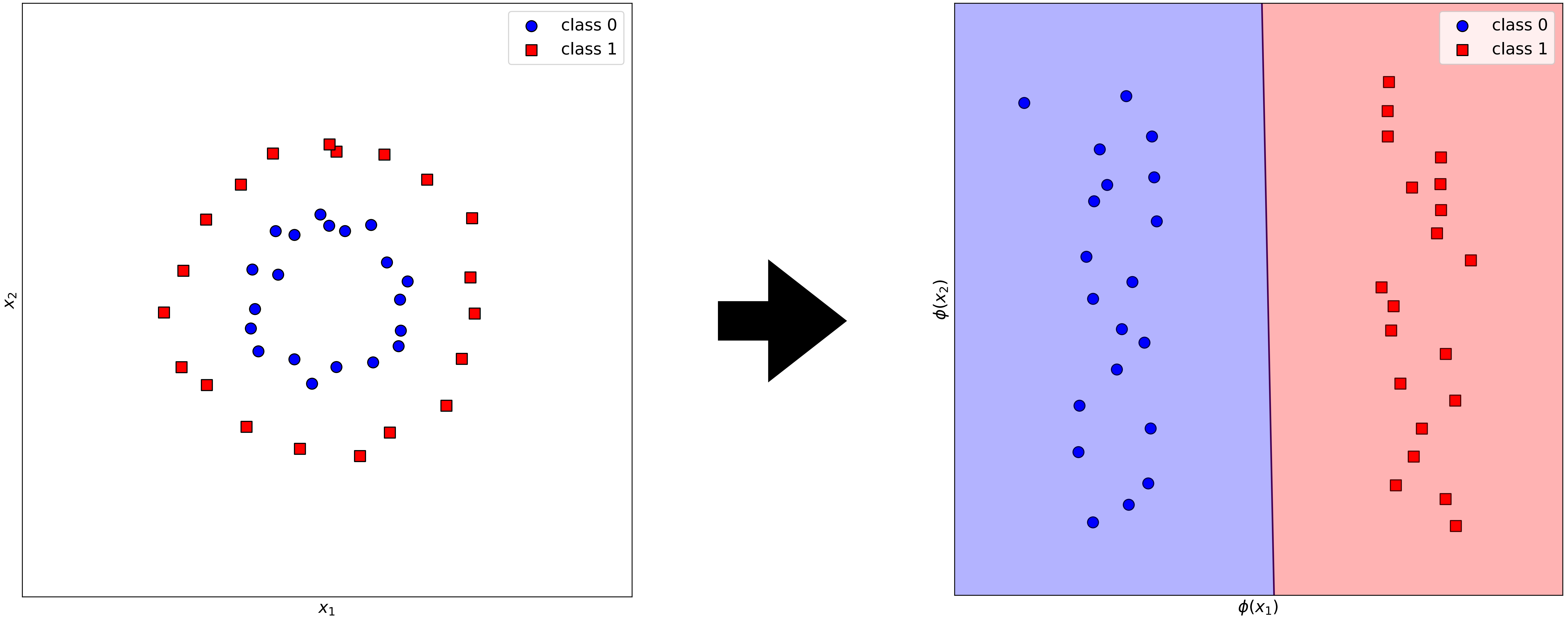


Feature transformations

- We can introduce some feature transformation ϕ to map our data into a space where it is linearly separable. We can then use $f(\mathbf{x}) = \mathbf{w}^\top \phi(\mathbf{x}) + b$
- We'll come back to this later in the course

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\phi(\mathbf{x}) = \begin{bmatrix} \|\mathbf{x}\| \\ \tan^{-1} \frac{x_2}{x_1} \end{bmatrix}^\top$$



Fitting a linear classifier

- For the classifier to be any good we learn the model parameters $\mathbf{w}, b$ using training data
- There are lots of ways to do this but they all largely boil down to minimising different loss functions that involve classifier scores $f(\mathbf{x})$ and labels y
- The loss functions rarely involve discrete predictions as the threshold function has a gradient of zero everywhere it is defined!
- We are going to cover logistic regression in detail and then look at some other approaches

Logistic Regression

First… treat classification as regression

- Consider a training set $\{(\mathbf{x}^{(n)}, y^{(n)})\}_{n=1}^{N}$ where $\mathbf{x} \in \mathbb{R}^D$ and $y \in \{0,1\}$
- Let's treat y as continuous $y \in \mathbb{R}$: *it just happens to be 0/1 for training data*
- We can use $f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} + b$ to predict this "continuous" label
- We could just minimise $L_{MSE} = \frac{1}{N} \sum_n (y^{(n)} - f(\mathbf{x}^{(n)}))^2$
- We can just use e.g. $\hat{y} = \begin{cases} 1 & \text{if } f(\mathbf{x}) > 0.5 \\ 0 & \text{if } f(\mathbf{x}) < 0.5 \end{cases}$ as our threshold function
- This is known as label regression. Our $f(\mathbf{x})$ isn't particularly meaningful

Logistic Regression

- Probabilities are meaningful as they quantify uncertainty
- We want to predict $p(y = 1 \mid \mathbf{x})$: the probability that $\mathbf{x}$ belongs to class 1
- We can't predict this with our linear model $f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} + b$ however
- This is because probabilities must lie between 0 and 1 and $f(\mathbf{x})$ is unbounded
- Let's instead predict an unbounded quantity that is related to $p(y = 1 \mid \mathbf{x})$

$$f(\mathbf{x}) = \log \frac{p(y = 1 \mid \mathbf{x})}{1 - p(y = 1 \mid \mathbf{x})}$$

The log-odds

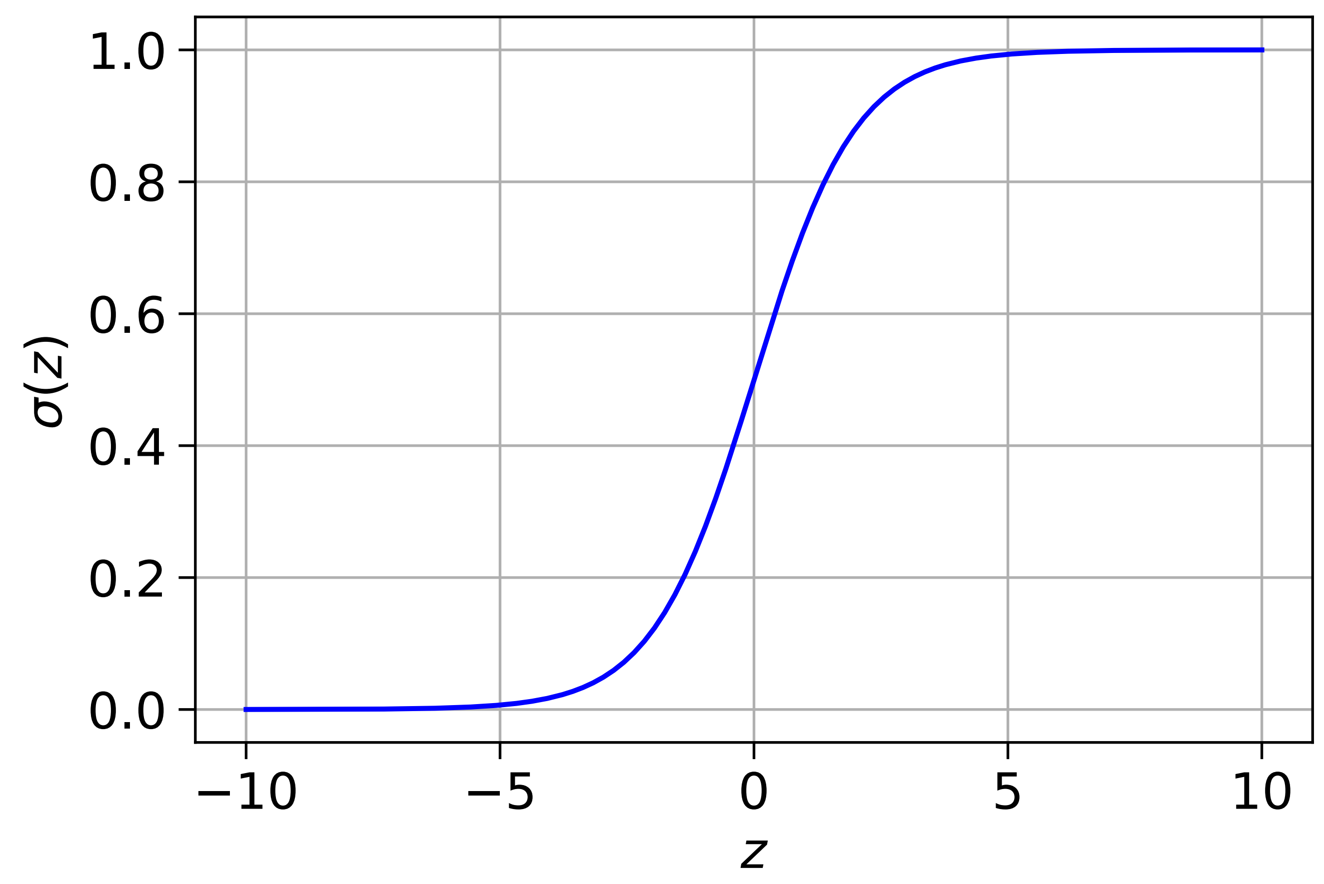
The sigmoid function

- In logistic regression our model predicts the log-odds for class 1

$$f(\mathbf{x}) = \log \frac{p(y=1\,|\,\mathbf{x})}{1 - p(y=1\,|\,\mathbf{x})}$$

- We can rearrange to express $p(y=1\,|\,\mathbf{x})$ in terms of log-odds

$$p(y=1\,|\,\mathbf{x}) = \frac{1}{1+e^{-f(\mathbf{x})}} = \sigma(f(\mathbf{x})) = \sigma(\mathbf{w}^\top \mathbf{x} + b)$$



σ is the sigmoid function.
It squashes numbers to be between 0 and 1

Discrete class predictions from log-odds

- We can convert log-odds to probabilities through $p(y = 1 \mid \mathbf{x}) = \sigma(f(\mathbf{x}))$
- It follows that $p(y = 0 \mid \mathbf{x}) = 1 - \sigma(f(\mathbf{x}))$ as there are only two classes
- What threshold function should we use to make a discrete class prediction $\hat{y}$?
- The obvious approach is to use $\hat{y} = \begin{cases} 1 & \text{if } p(y = 1 \mid \mathbf{x}) \geq 0.5 \\ 0 & \text{if } p(y = 1 \mid \mathbf{x}) < 0.5 \end{cases}$
- $\sigma(\mathbf{w}^\top \mathbf{x} + b) = 0.5$ when $\mathbf{w}^\top \mathbf{x} + b = 0$ which is a **hyperplane**
- We can rewrite the above as $\hat{y} = \begin{cases} 1 & \text{if } f(\mathbf{x}) \geq 0 \\ 0 & \text{if } f(\mathbf{x}) < 0 \end{cases}$

Maximum likelihood estimation

- We can write $p(y \mid \mathbf{x}) = \sigma(f(\mathbf{x}))^y (1 - \sigma(f(\mathbf{x})))^{1-y}$

- We can then write an expression for the likelihood of our data

$$\prod_n p(y^{(n)} \mid \mathbf{x}^{(n)}) = \prod_n \sigma(f(\mathbf{x}^{(n)}))^{y^{(n)}} (1 - \sigma(f(\mathbf{x}^{(n)})))^{1-y^{(n)}}$$

- Maximising likelihood is the same as minimising negative log-likelihood (divided by the number of data points)

$$\mathrm{NLL}(\mathbf{w}, b) = -\frac{1}{N} \sum_n \left[y^{(n)} \log \sigma(f(\mathbf{x}^{(n)})) + (1 - y^{(n)}) \log(1 - \sigma(f(\mathbf{x}^{(n)}))) \right]$$

NLL is the log loss

$$\text{NLL}(\mathbf{w}, b) = -\frac{1}{N}\sum_n \left[y^{(n)} \log \sigma(f(\mathbf{x}^{(n)})) + (1 - y^{(n)})\log(1 - \sigma(f(\mathbf{x}^{(n)})))\right]$$

- We can write $p(y = 1 \mid \mathbf{x}) = \sigma(f(\mathbf{x}^{(n)})$ as $p^{(n)}$ to express this more succinctly:

$$L_{log} = -\frac{1}{N}\sum_n \left[y^{(n)} \log p^{(n)} + (1 - y^{(n)})\log(1 - p^{(n)})\right]$$

- It is also know as the logistic loss, or the cross-entropy loss. Minimising it is performing maximum likelihood estimation (MLE)

- Cross-entropy is a quantity that crops up in information theory. It measures how much the probabilities produced by our model differ from the true probabilities (so low = good)

Log loss

$$L_{log} = -\frac{1}{N}\sum_{n}\left[y^{(n)}\log p^{(n)} + (1 - y^{(n)})\log(1 - p^{(n)})\right]$$

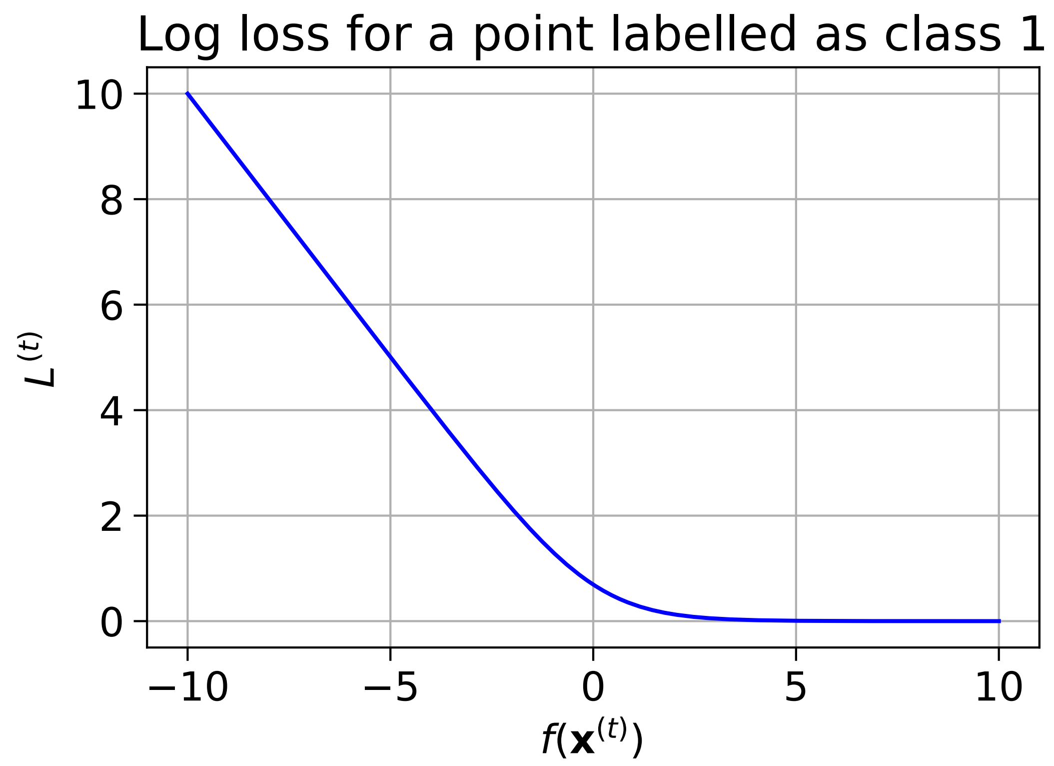
- This loss is convex for a linear classifier
- We can use a gradient-based optimiser to solve $\underset{\mathbf{w},b}{\text{minimise}}\ L_{log}$ using

$$\nabla_{\mathbf{w}} L_{log} = -\frac{1}{N}\sum_{n}\left(y^{(n)} - p^{(n)}\right)\mathbf{x}^{(n)}$$

$$\nabla_{b} L_{log} = -\frac{1}{N}\sum_{n}\left(y^{(n)} - p^{(n)}\right)$$

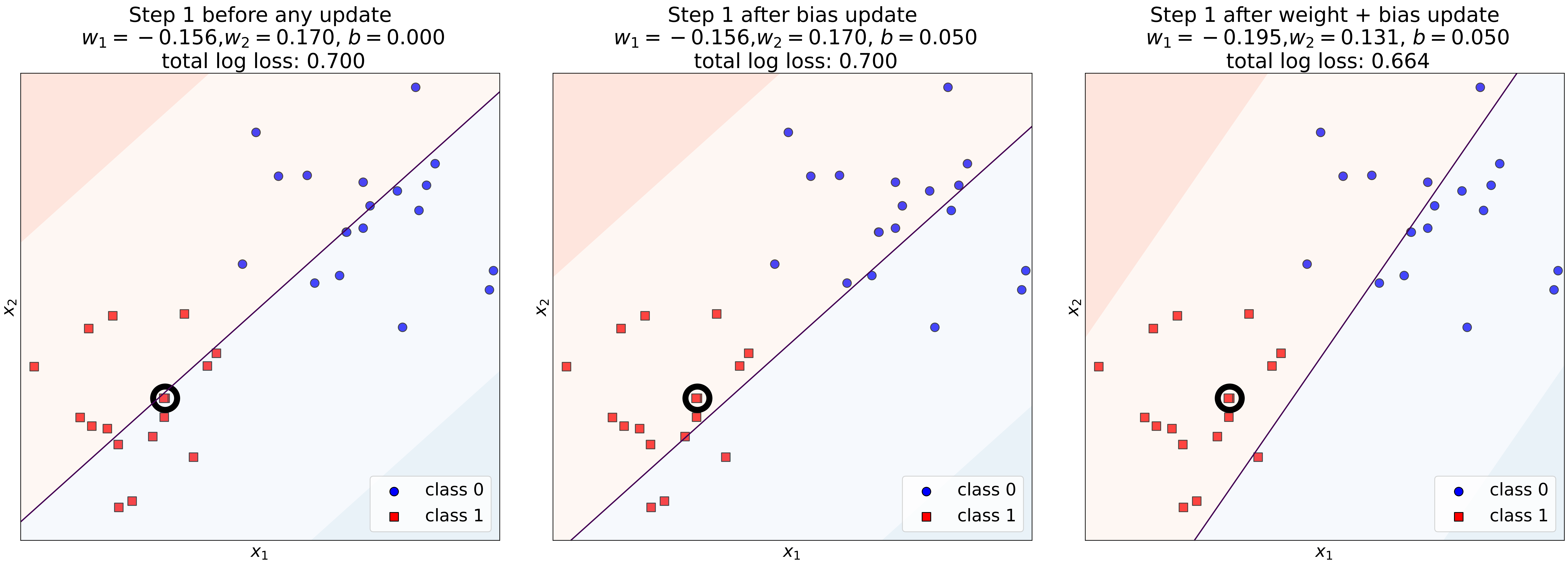
Stochastic Gradient descent (SGD) algorithm

- Goal: We want to minimise the training loss of our model on $\mathfrak{D} = \{(\mathbf{x}^{(n)}, y^{(n)})\}_{n=1}^{N}$
- We can write the loss as an average of per example losses $\frac{1}{N}\sum_{n} L(y^{(n)}, \mathbf{x}^{(n)}, \mathbf{w}, b) = \frac{1}{N}\sum_{n} L^{(n)}$
- Initialise $\mathbf{w}$ and b
- For epoch in range(E):
    - Shuffle $\mathfrak{D}$
    - For n in range(N):
        - Compute $\nabla_{\mathbf{w}} L^{(n)}$ and $\nabla_{b} L^{(n)}$
        - Update $\mathbf{w} \leftarrow \mathbf{w} - \alpha \nabla_{\mathbf{w}} L^{(n)}$ and $b \leftarrow b - \alpha \nabla_{b} L^{(n)}$

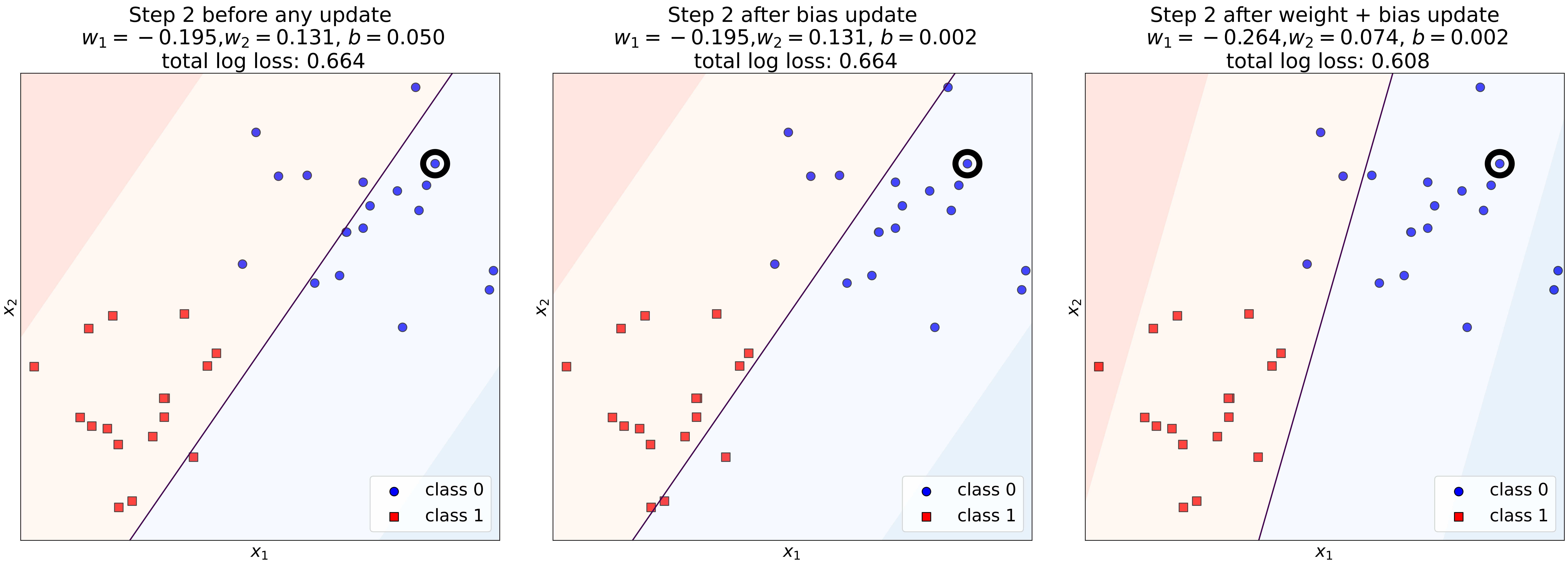


This is what sklearn does. There are lots of variants.
See https://sebastianraschka.com/faq/docs/sgd-methods.html

SGD Update 1

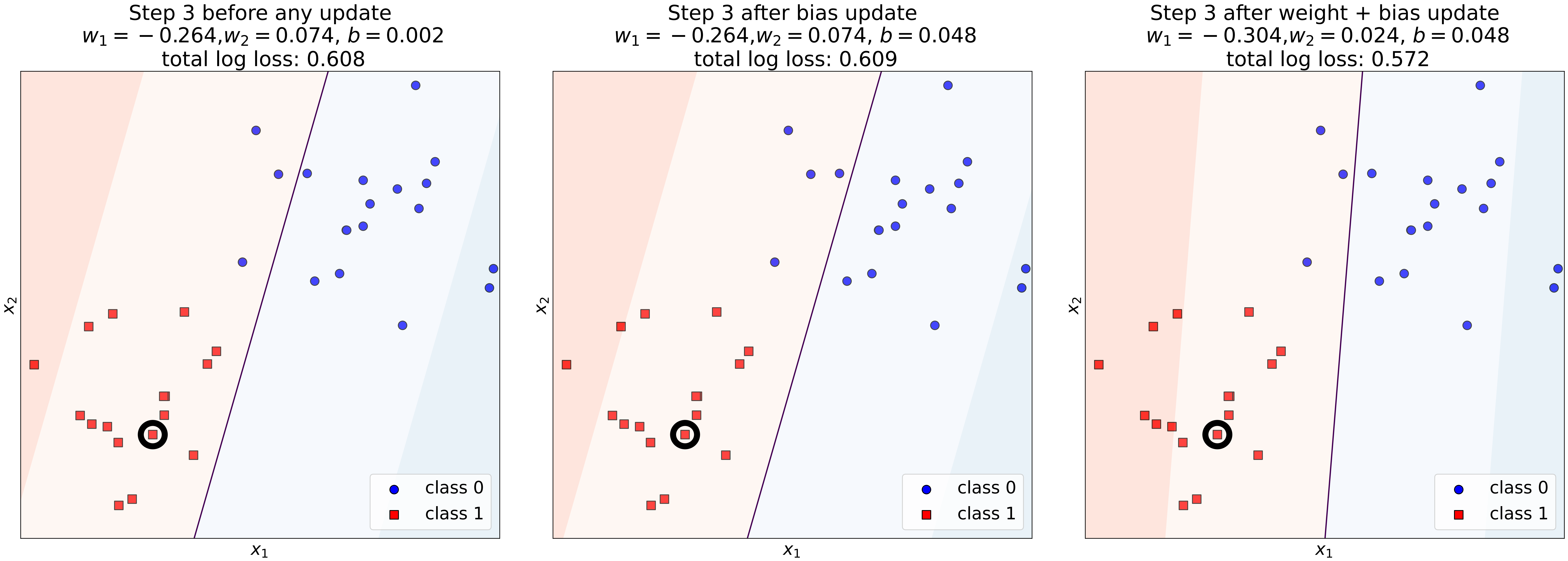

Step 1 before any update
$w_1 = -0.156, w_2 = 0.170$, $b = 0.000$
total log loss: 0.700
x_2
x_1
class 0
class 1
Step 1 after bias update
$w_1 = -0.156, w_2 = 0.170$, $b = 0.050$
total log loss: 0.700
x_2
x_1
class 0
class 1
Step 1 after weight + bias update
$w_1 = -0.195, w_2 = 0.131$, $b = 0.050$
total log loss: 0.664
x_2
x_1
class 0
class 1

SGD Update 2



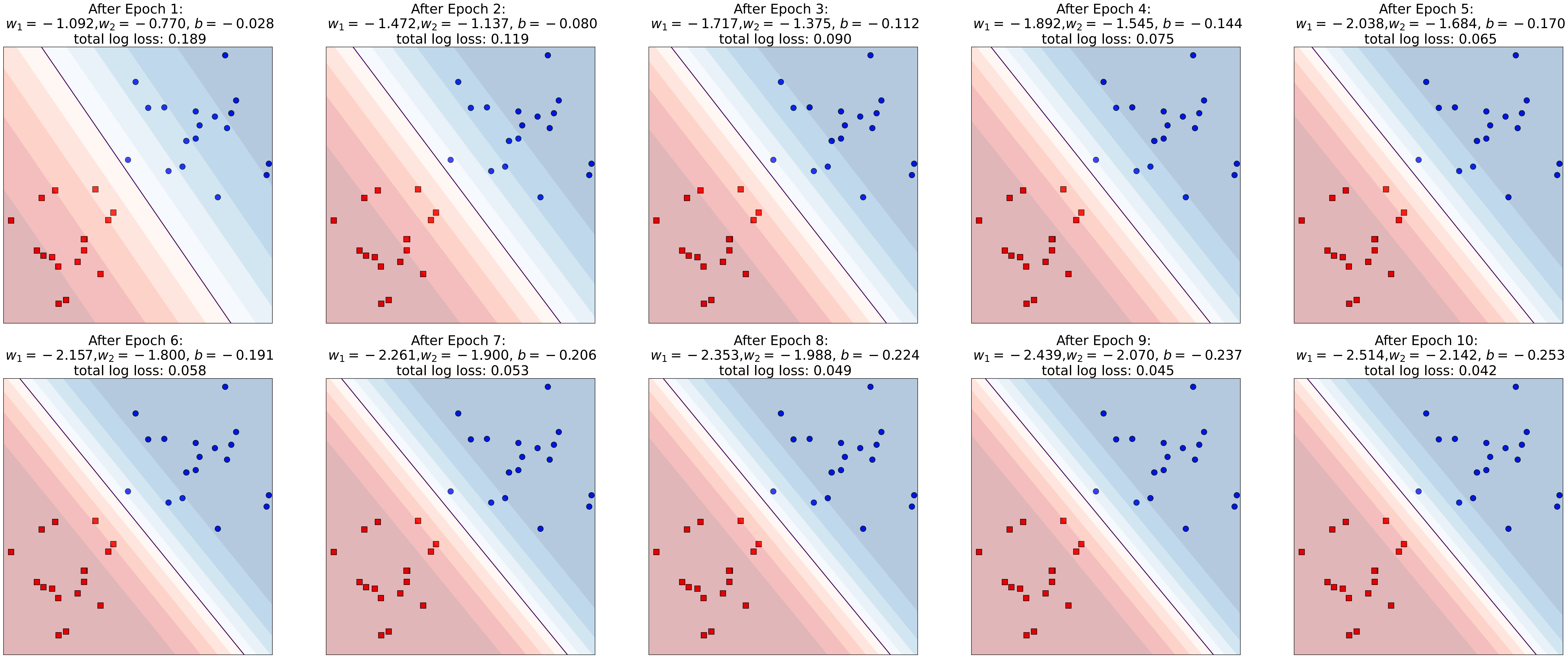
Step 2 before any update
$w_1 = -0.195, w_2 = 0.131, b = 0.050$
total log loss: 0.664
x_1
x_2
class 0
class 1
Step 2 after bias update
$w_1 = -0.195, w_2 = 0.131, b = 0.002$
total log loss: 0.664
x_1
x_2
class 0
class 1
Step 2 after weight + bias update
$w_1 = -0.264, w_2 = 0.074, b = 0.002$
total log loss: 0.608
x_1
x_2
class 0
class 1

SGD Update 3

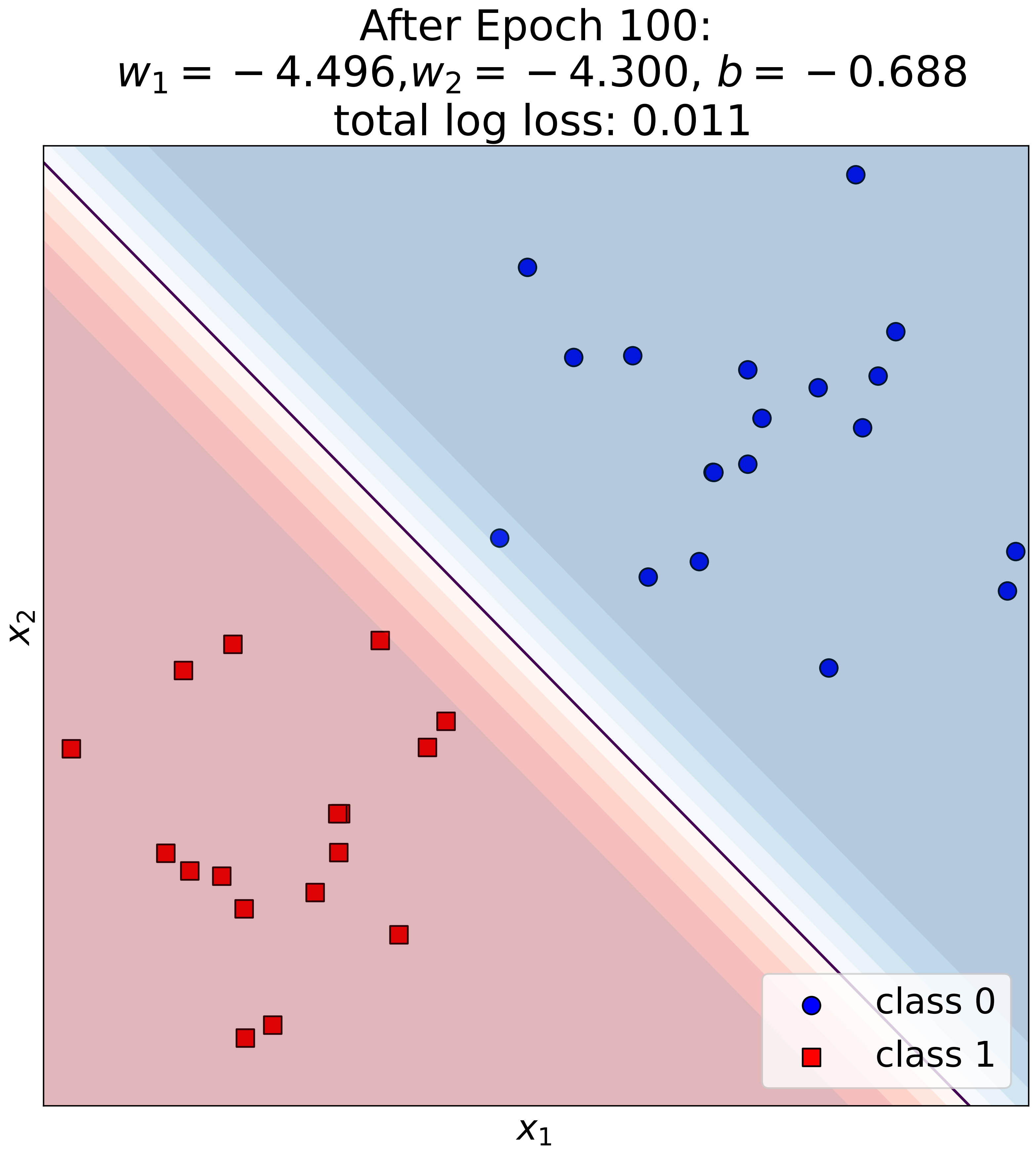


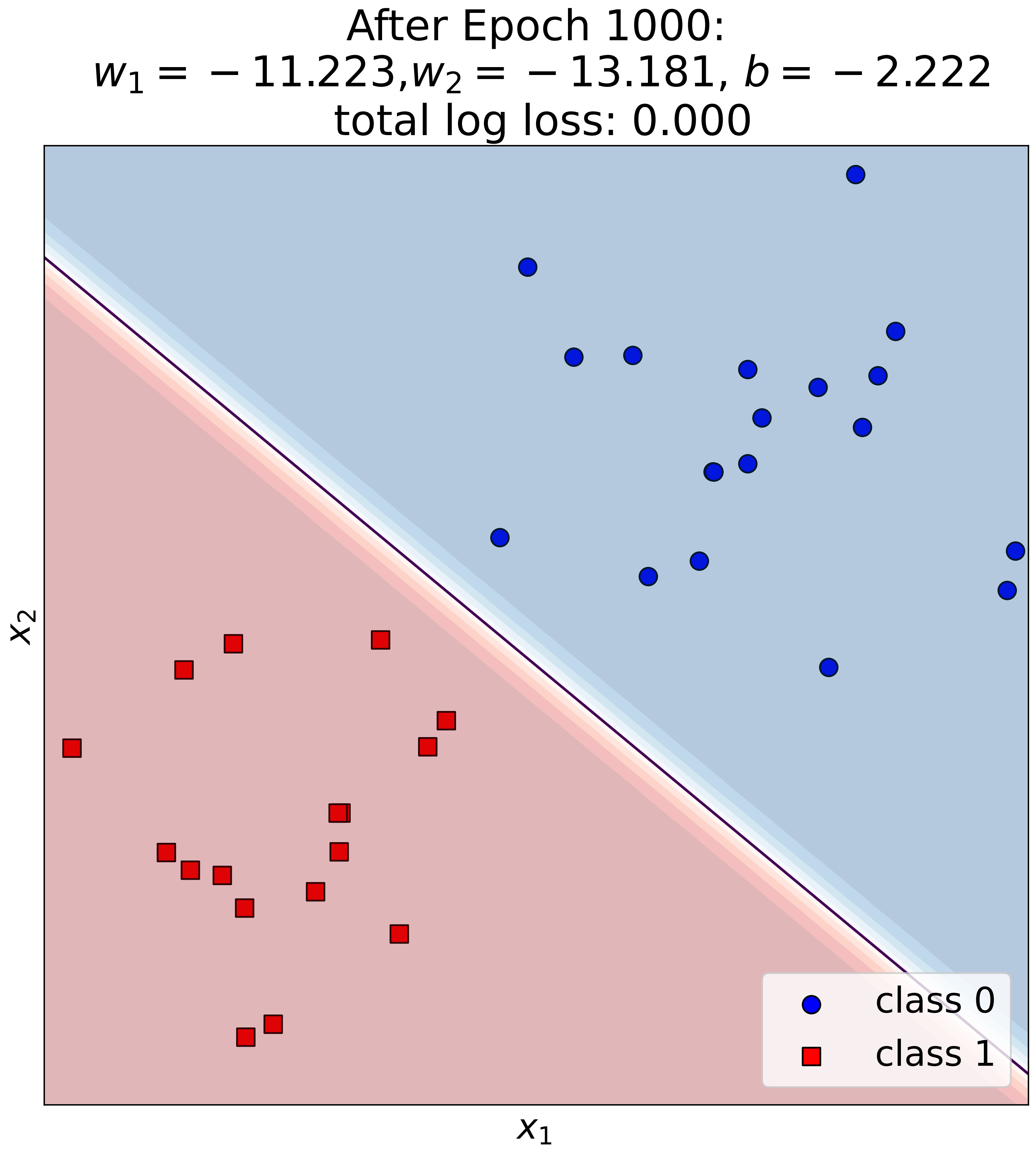
Step 3 before any update
$w_1 = -0.264, w_2 = 0.074,\ b = 0.002$
total log loss: 0.608
Step 3 after bias update
$w_1 = -0.264, w_2 = 0.074,\ b = 0.048$
total log loss: 0.609
Step 3 after weight + bias update
$w_1 = -0.304, w_2 = 0.024,\ b = 0.048$
total log loss: 0.572
x_1
x_2
class 0
class 1

SGD for 10 epochs



“Don’t get cocky!”





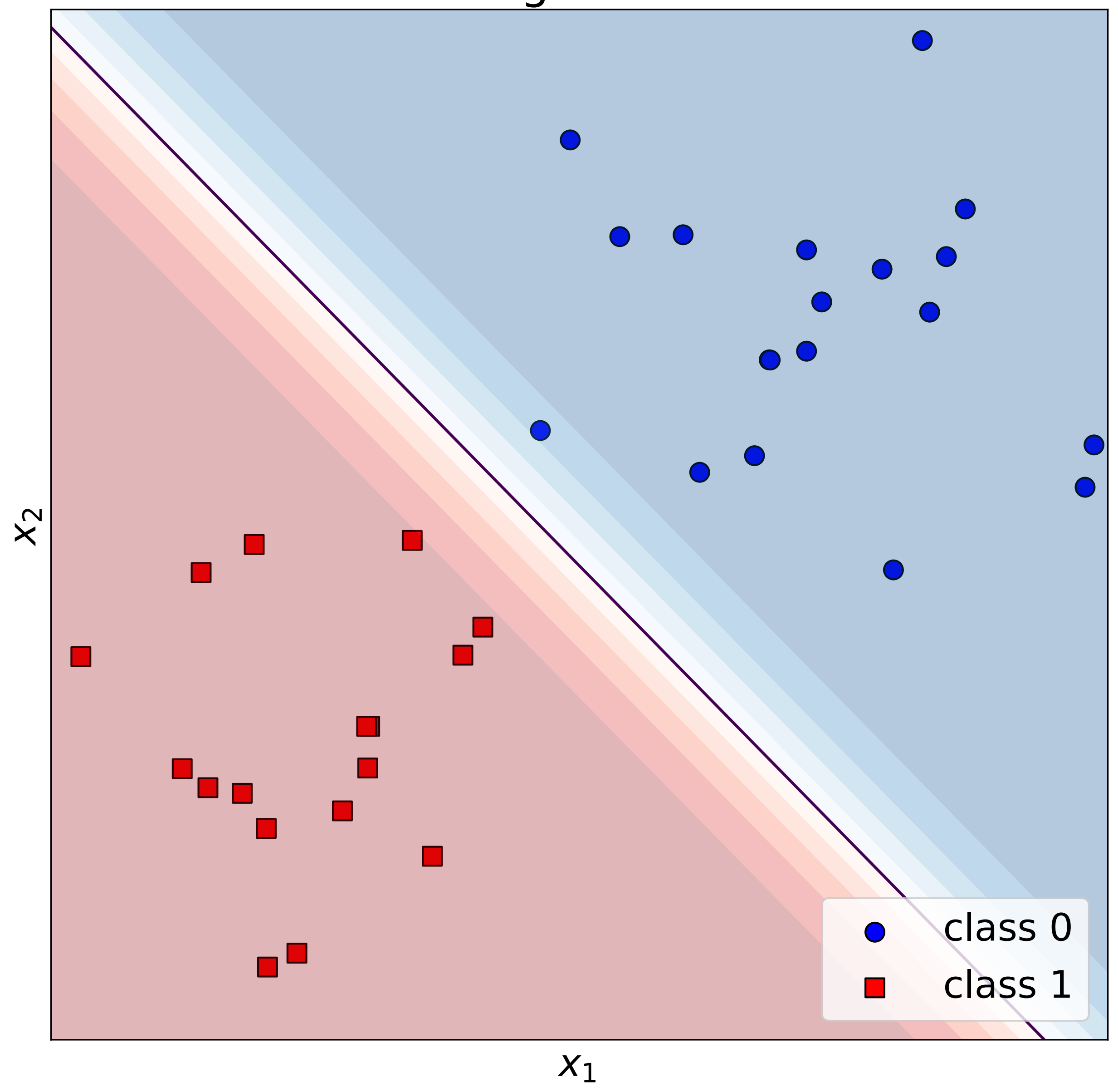
Regularisation

- We are seeing a model that is far too confident near the boundary. It is overfitting to the training data
- And look… its weights are large!
- We can add regularisation as we did for regression

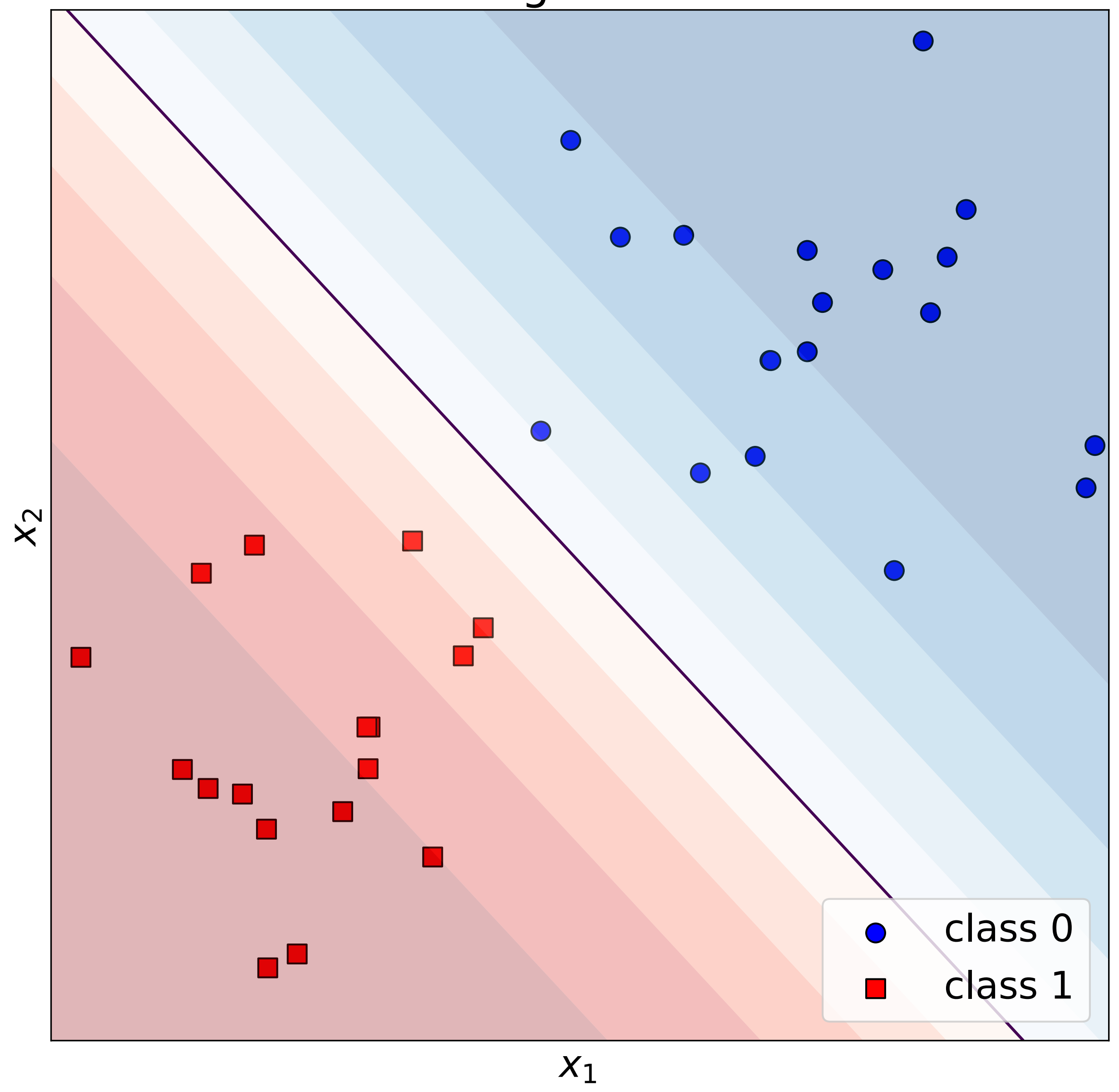
$$L_{total} = \underbrace{L_{log}}_{classification} + \underbrace{\frac{\lambda}{2}\|\mathbf{w}\|^2}_{regularisation}$$

We can use the validation set to find the optimal λ

After Epoch 100:
$w_1 = -4.496, w_2 = -4.300,\ b = -0.688$
total log loss: 0.011

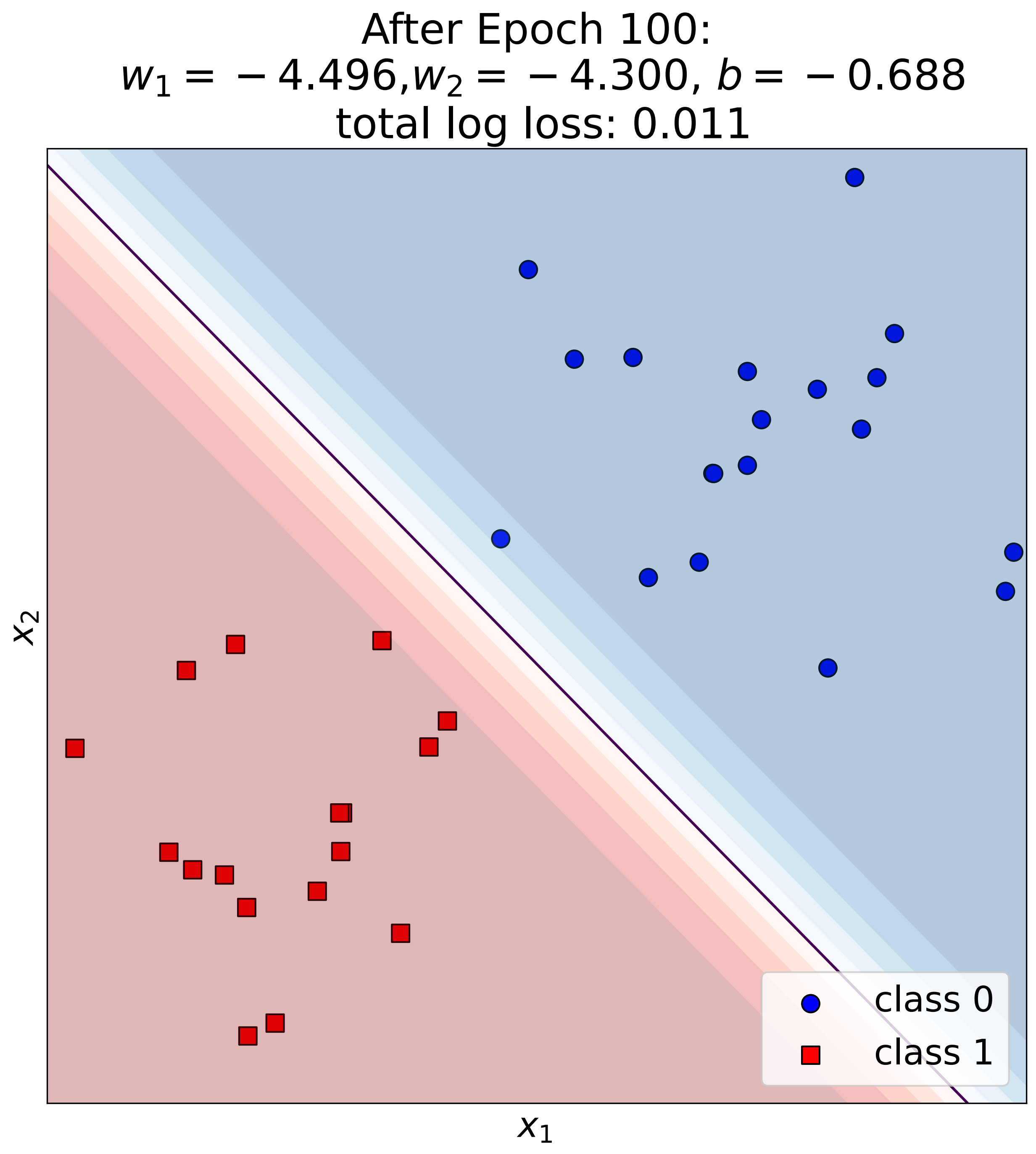


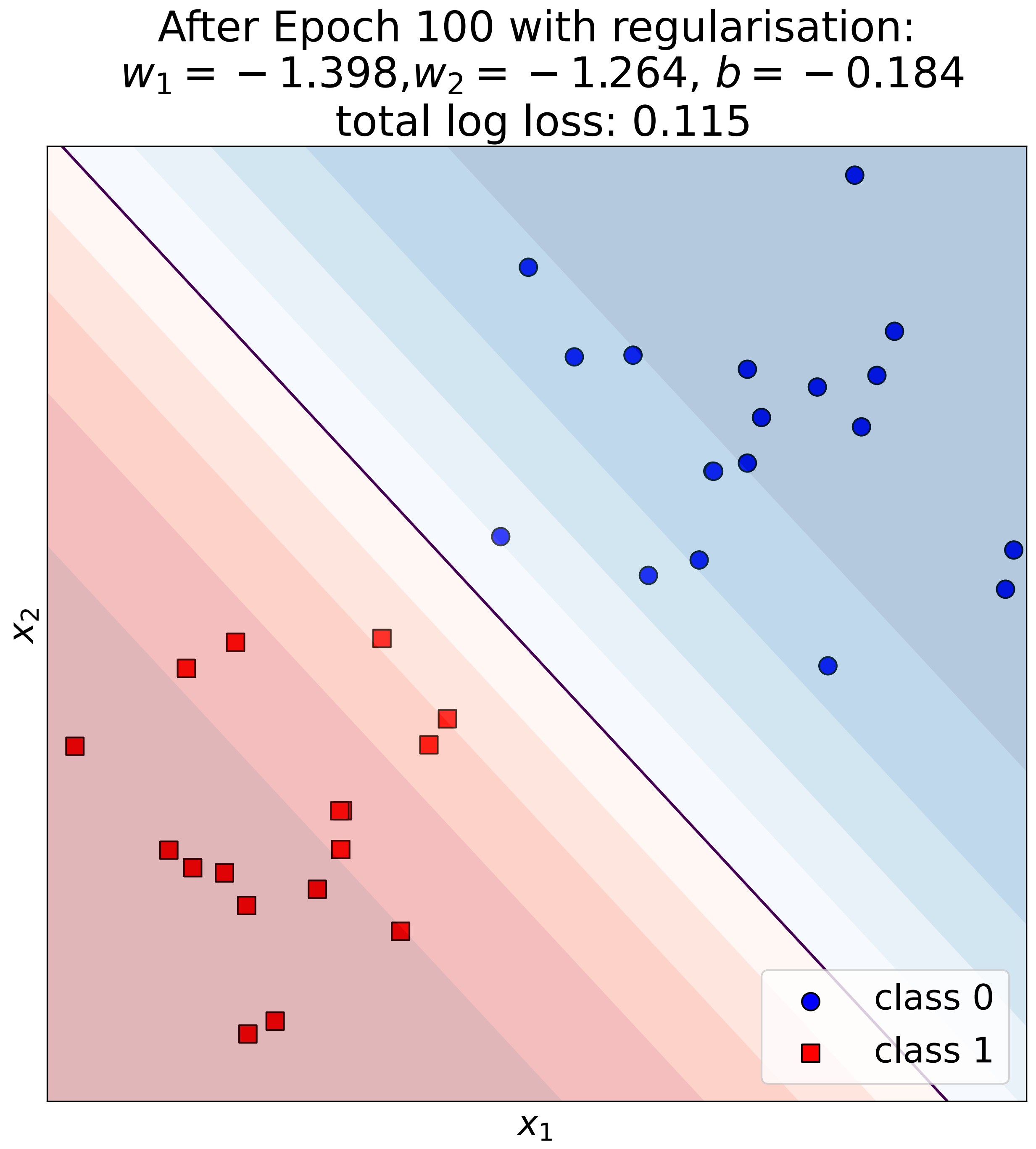
After Epoch 100 with regularisation:
$w_1 = -1.398, w_2 = -1.264,\ b = -0.184$
total log loss: 0.115



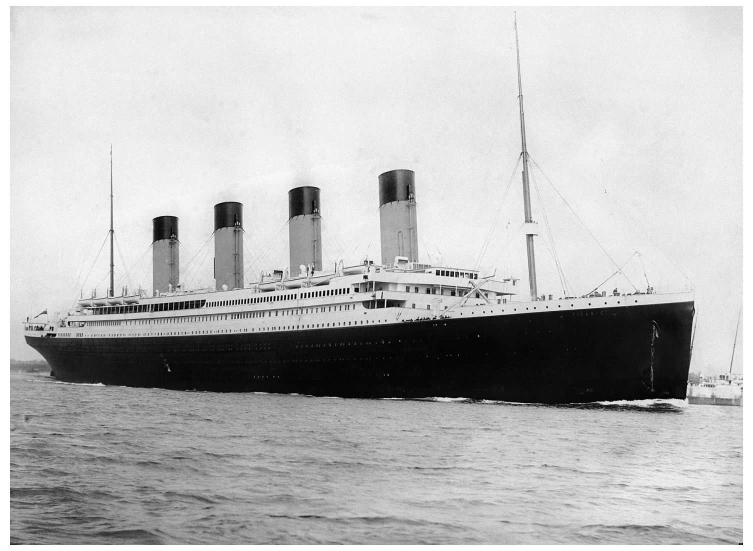
Wait, why did that help?

- Consider the line of points that would correspond to some log-odds k
- This is just the line $k = \mathbf{w}^\top \mathbf{x} + b$
- The decision boundary is the line $0 = \mathbf{w}^\top \mathbf{x} + b$
- The distance between these two parallel lines is $\dfrac{k}{\|\mathbf{w}\|}$
- Regularising increase this distance, making the model less confident near the boundary





Titanic Dataset



- We can use historical data about passengers to learn a linear classifier to predict survival using logistic regression

Pclass	Sex	Age	SibSp	Parch	Fare	Survived
3	0	22.0	1	0	7.2500	0
1	1	38.0	1	0	71.2833	1
3	1	26.0	0	0	7.9250	1
1	1	35.0	1	0	53.1000	1
3	0	35.0	0	0	8.0500	0
...	...	...	...	...	...	...
3	1	39.0	0	5	29.1250	0
2	0	27.0	0	0	13.0000	0
1	1	19.0	0	0	30.0000	1
1	0	26.0	0	0	30.0000	1
3	0	32.0	0	0	7.7500	0

For "Sex", *male* has been mapped to 0 and *female* to 1 arbitrarily

- If we standardise data then the weights we learn are interpretable

$$\mathbf{w} = [\underset{\text{Pclass}}{-0.97} \quad \underset{\text{Sex}}{1.27} \quad \underset{\text{Age}}{-0.52} \quad \underset{\text{SibSp}}{-0.27} \quad \underset{\text{Parch}}{-0.03} \quad \underset{\text{Fare}}{0.16}]^\top$$

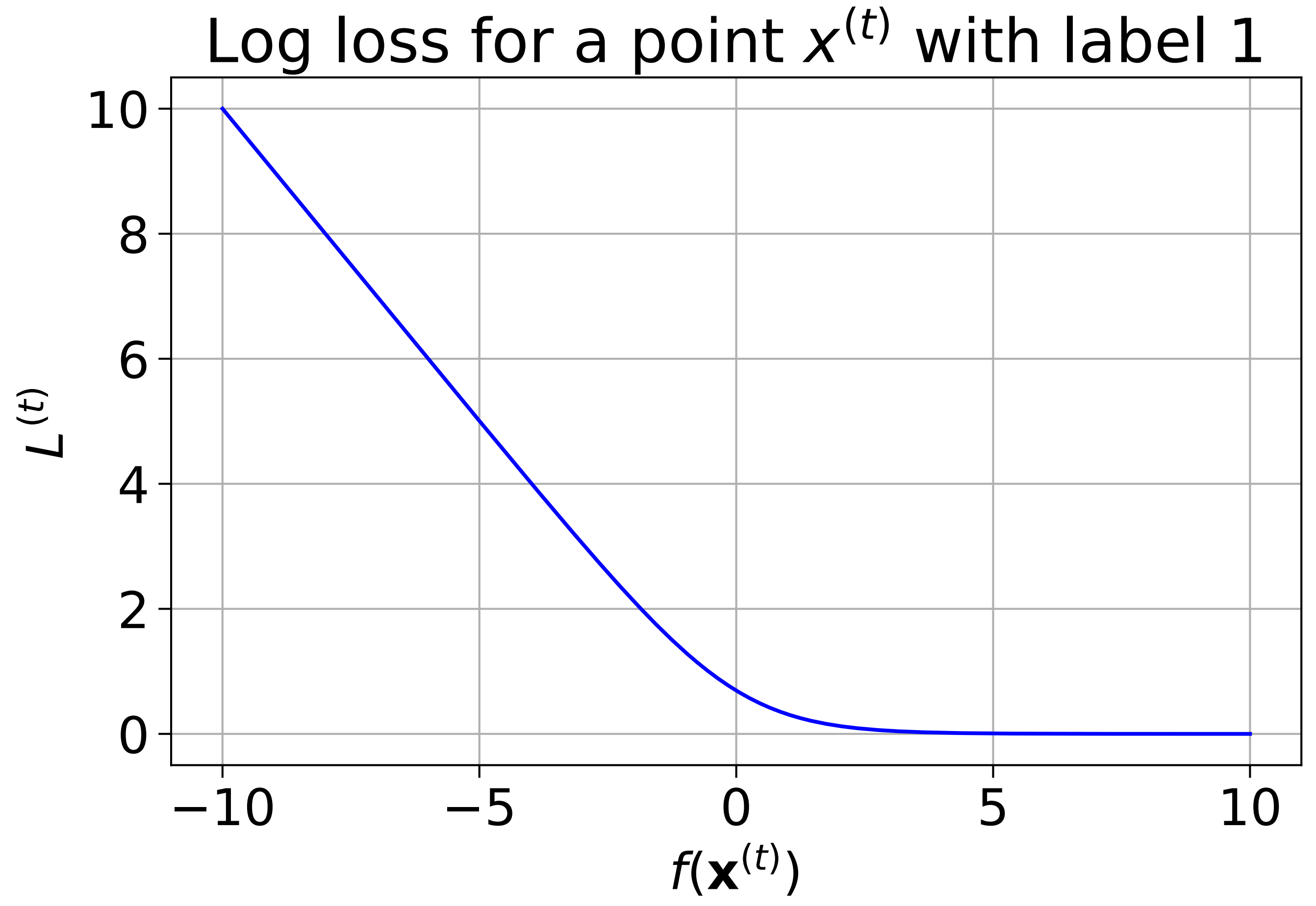
- Survival more probable for people who are in first class, female, young

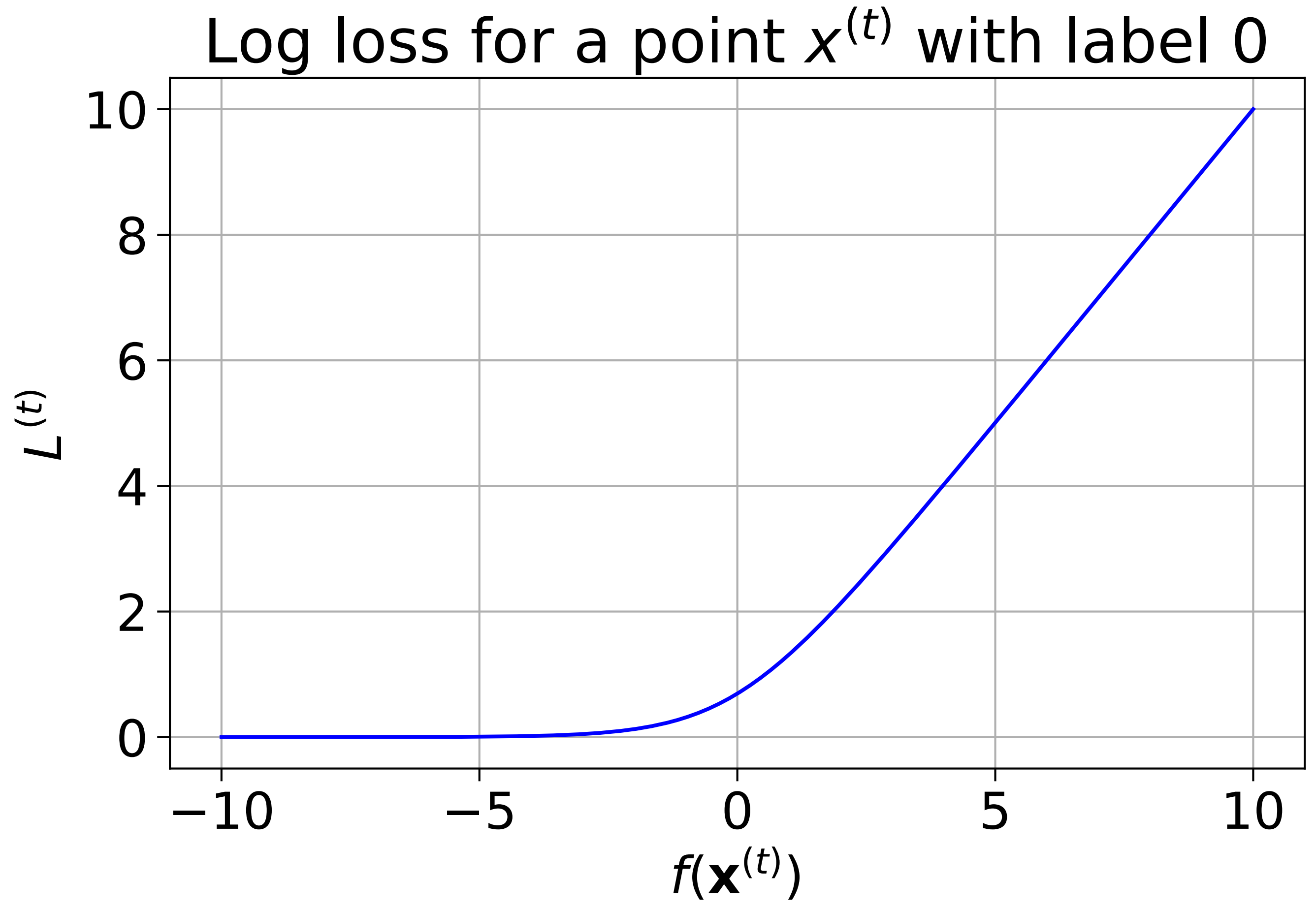
Gets 80% on held-out data so is a reasonable model

Logistic regression in short

- Given a linear model $f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} + b$, logistic regression is solving $\underset{\mathbf{w},b}{\text{minimise}}\ L_{log}$

$$L_{log} = -\frac{1}{N}\sum_n \left[y^{(n)} \log \sigma(f(\mathbf{x}^{(n)})) + (1 - y^{(n)}) \log(1 - \sigma(f(\mathbf{x}^{(n)}))\right]$$



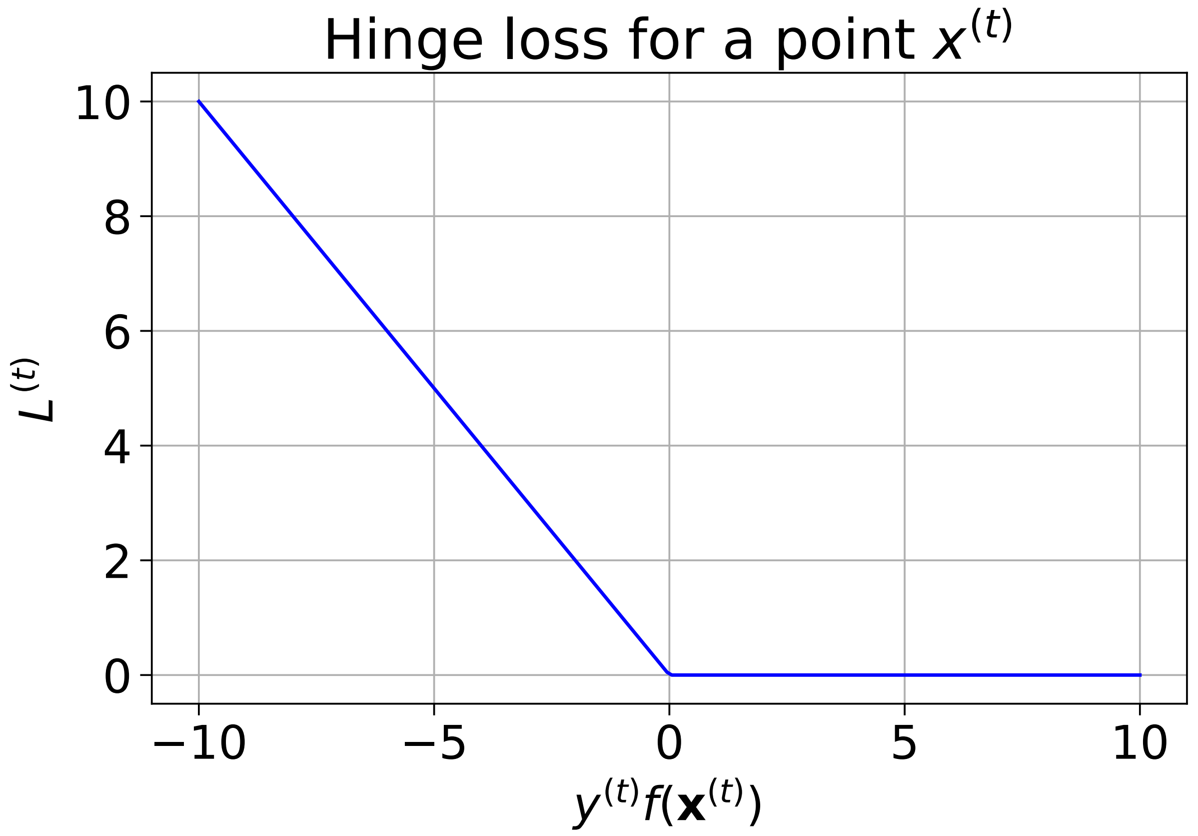


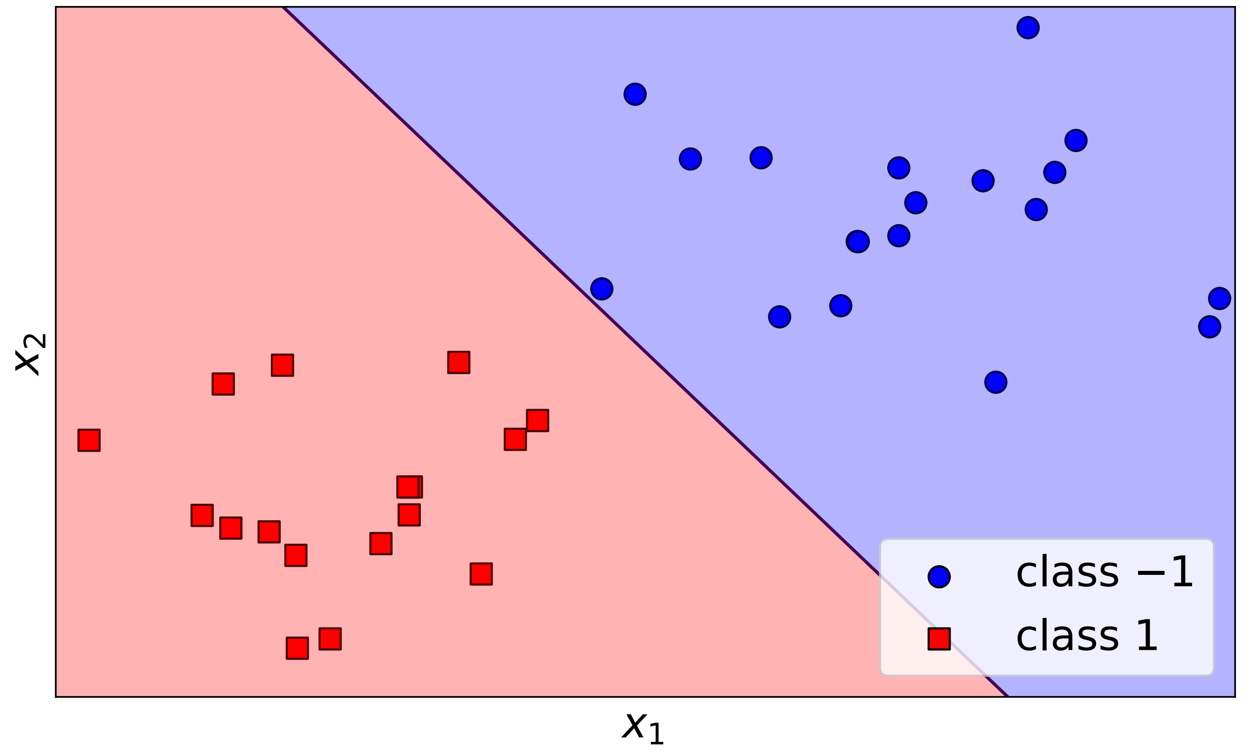
- There are other methods that boil down to minimising different losses

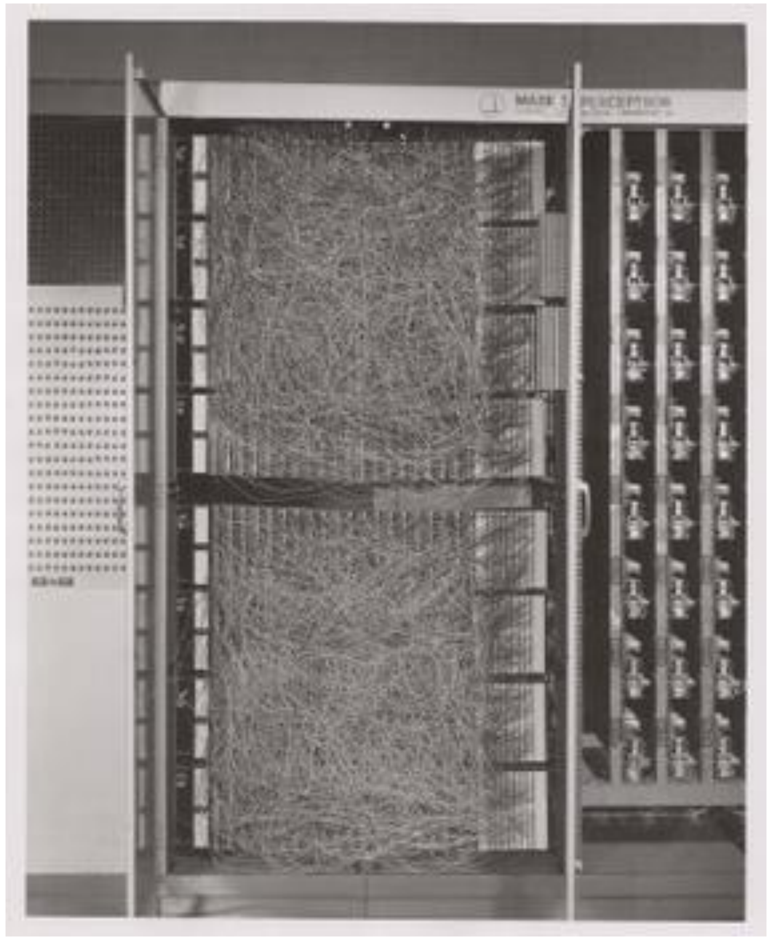
Perceptron learning

- Consider a training set $\{(\mathbf{x}^{(n)}, y^{(n)})\}_{n=1}^{N}$ where $\mathbf{x} \in \mathbb{R}^D$ and $y \in \{-1,1\}$
- Given a linear model $f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} + b$ and threshold function $\hat{y} = \begin{cases} 1 & \text{if } f(\mathbf{x}) \geq 0 \\ -1 & \text{if } f(\mathbf{x}) < 0 \end{cases}$ the perceptron learning algorithm is equivalent to solving $\underset{\mathbf{w},b}{\text{minimise}}\ L_{hinge}$

$$L_{hinge} = \frac{1}{N}\sum_n \max\left(0, -y^{(n)} f(\mathbf{x}^{(n)})\right)$$





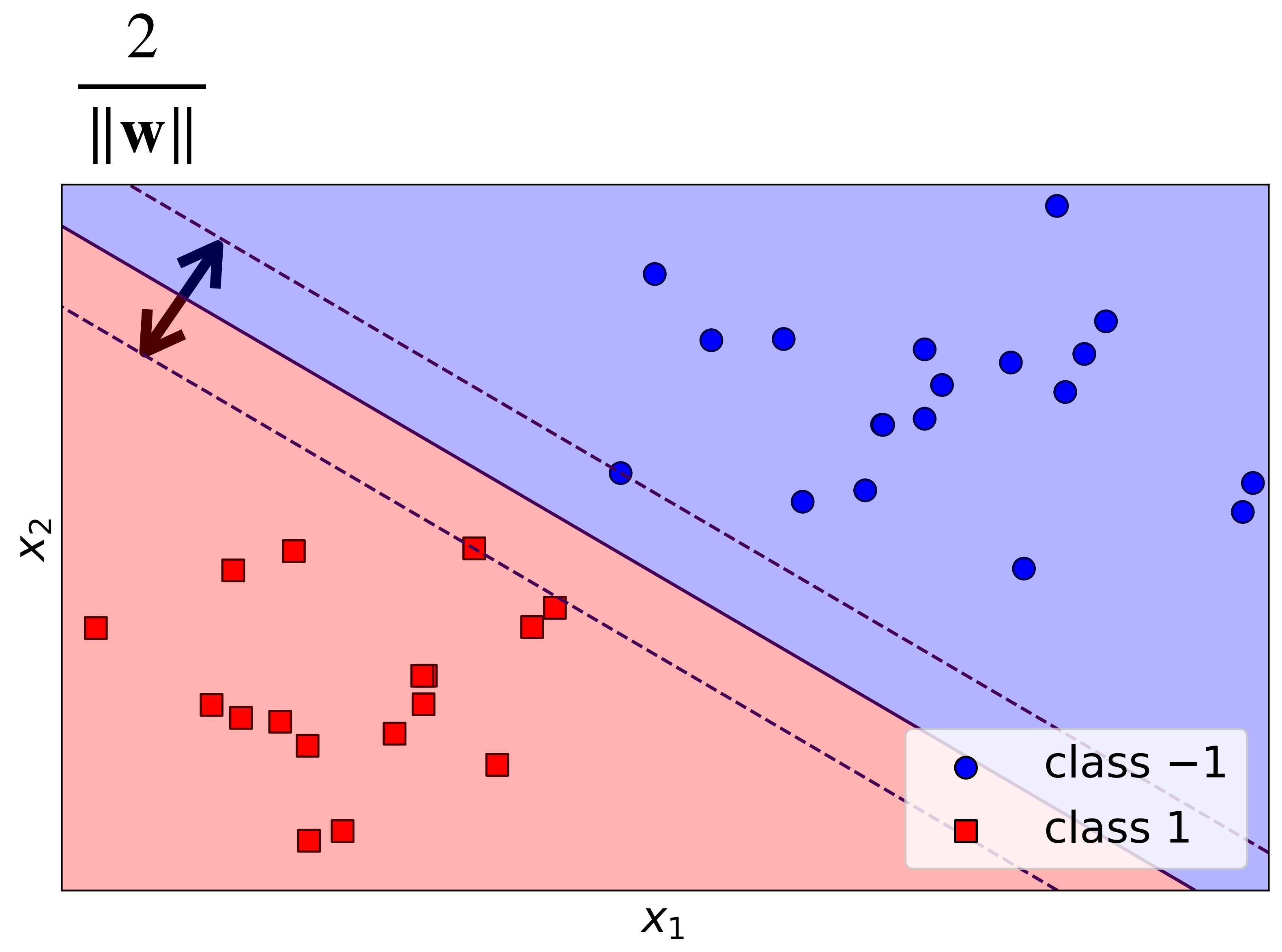


https://en.wikipedia.org/wiki/Perceptron

Support Vector Machines (SVMs)

- Consider a training set $\{(\mathbf{x}^{(n)}, y^{(n)})\}_{n=1}^{N}$ where $\mathbf{x} \in \mathbb{R}^D$ and $y \in \{-1,1\}$
- Given a linear model $f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} + b$ and threshold function $\hat{y} = \begin{cases} 1 & \text{if } f(\mathbf{x}) \geq 0 \\ -1 & \text{if } f(\mathbf{x}) < 0 \end{cases}$
(linear) SVM learning is equivalent to solving $\underset{\mathbf{w},b}{\text{minimise}}\ L_{SVM}$

$$L_{SVM} = \frac{1}{2}\|\mathbf{w}\|^2 + C\sum_n \max\left(0, 1 - y^{(n)} f(\mathbf{x}^{(n)})\right)$$



SVMs are maximum margin classifiers

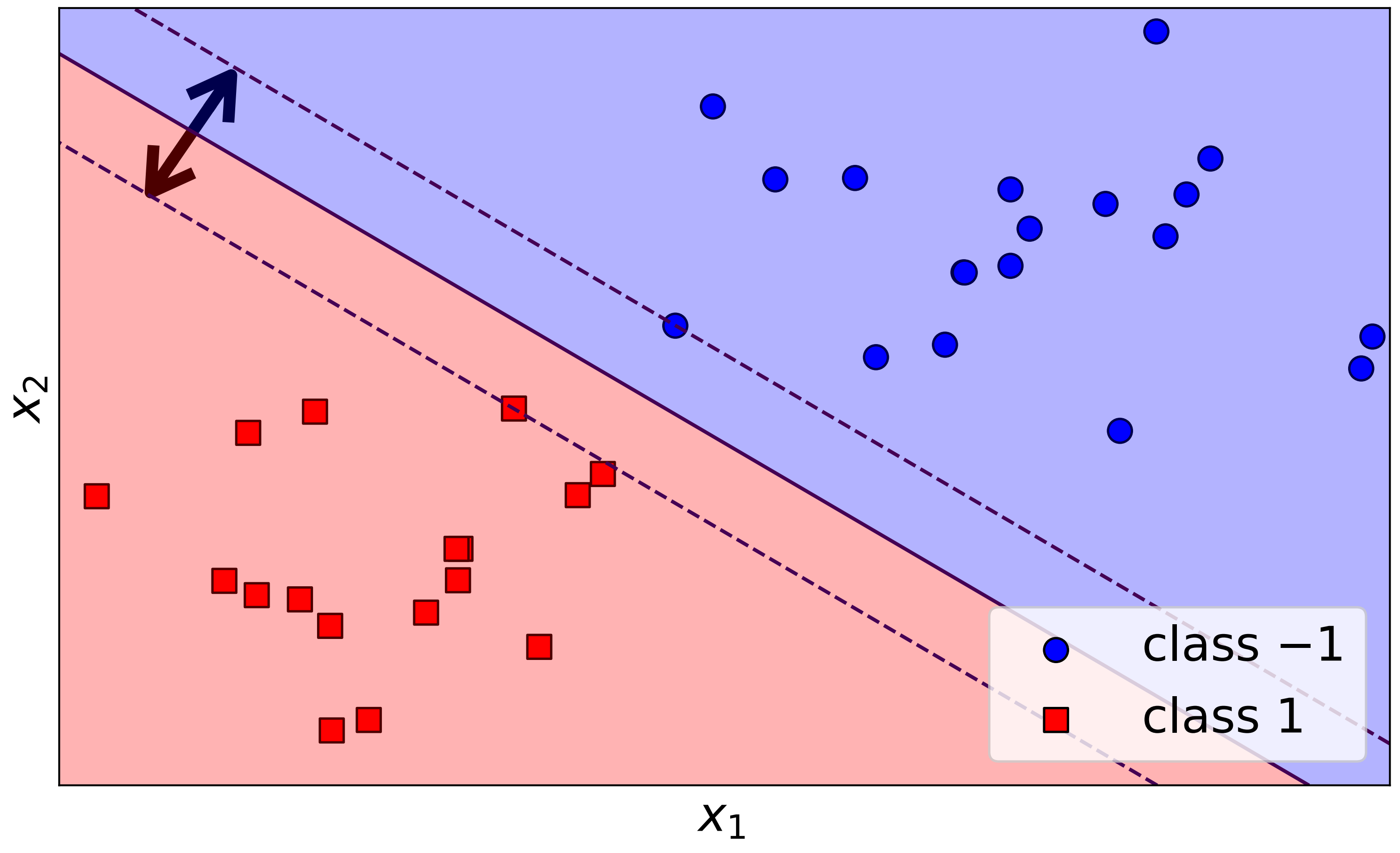
$$L_{SVM} = \frac{1}{2}\|\mathbf{w}\|^2 + C\sum_n \max\left(0, 1 - y^{(n)} f(\mathbf{x}^{(n)})\right)$$

If we define the margin as the region where $|f(\mathbf{x})| < 1$ then:

- The first term in L_{SVM} is small when the margin is big
- The second term in L_{SVM} is small when points don't live in the margin
- C is a hyperparameter that controls the relative importance of these terms

Logistic regression + sufficient regularisation also gives a large margin

$\frac{2}{\|\mathbf{w}\|}$



Multinomial Logistic Regression

Multi-class classification with linear classifiers

Now consider the multi-class scenario $\{(\mathbf{x}^{(n)}, y^{(n)})\}_{n=1}^{N}$ with $y \in \mathbb{Z}^{+}_{<K} = \{0,1,\ldots,K-1\}$.

There are three different approaches to solving this:

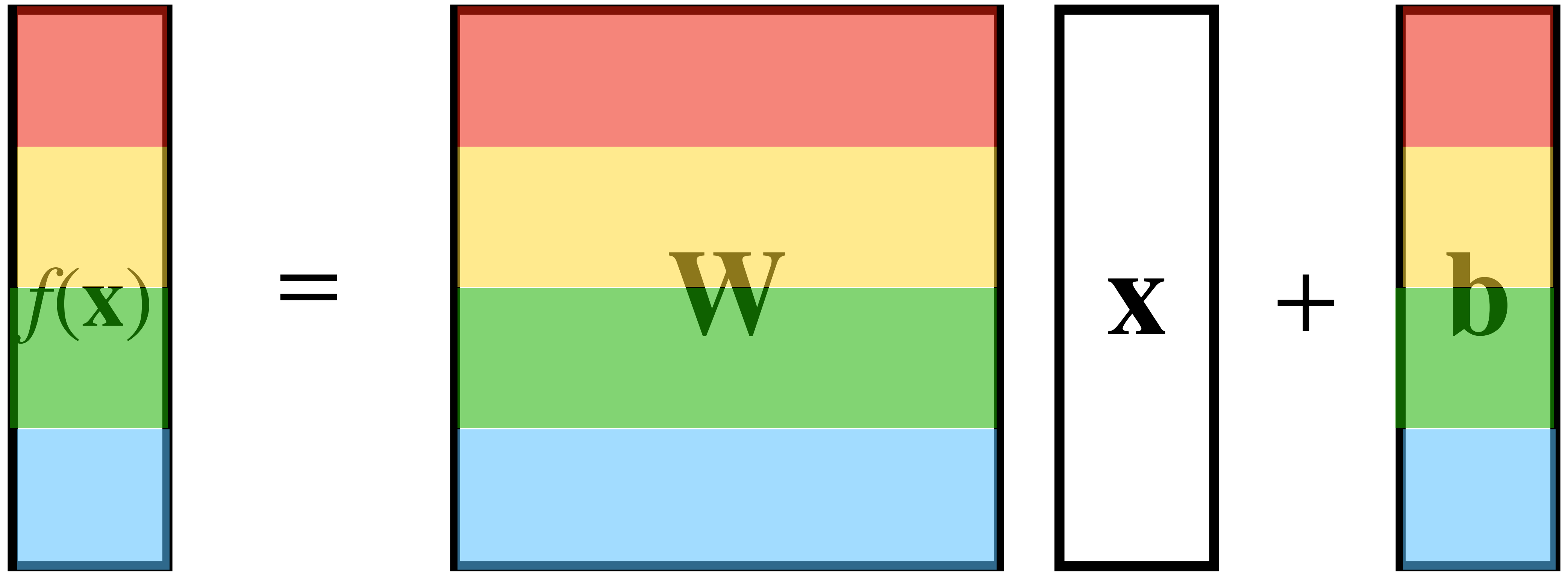
1. We could learn K one-vs-rest classifiers: $f_0(\mathbf{x}), f_1(\mathbf{x}), \ldots, f_{K-1}(\mathbf{x})$ and classify points according to the highest score

2. We could learn $(K(K-1))/2$ one-vs-one classifiers and classify points according to the majority vote

3. We could make our linear model output a **vector** where each element is a score for a different class and select the class with the highest score

A multi-class linear model

- In the binary case $f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} + b$ where $\mathbf{x} \in \mathbb{R}^D$ and $f(\mathbf{x}) \in \mathbb{R}$
- For our model to output a score for each of K classes we can:
    1. Replace the vector $\mathbf{w} \in \mathbb{R}^D$ with a matrix $\mathbf{W} \in \mathbb{R}^{K \times D}$
    2. Replace the bias vector $b \in \mathbb{R}$ with a vector $\mathbf{b} \in \mathbb{R}^K$
- This gives us $f(\mathbf{x}) = \mathbf{W}\mathbf{x} + \mathbf{b}$ where $f(\mathbf{x}) \in \mathbb{R}^K$



Like having K models side-by-side

Multinomial logistic regression

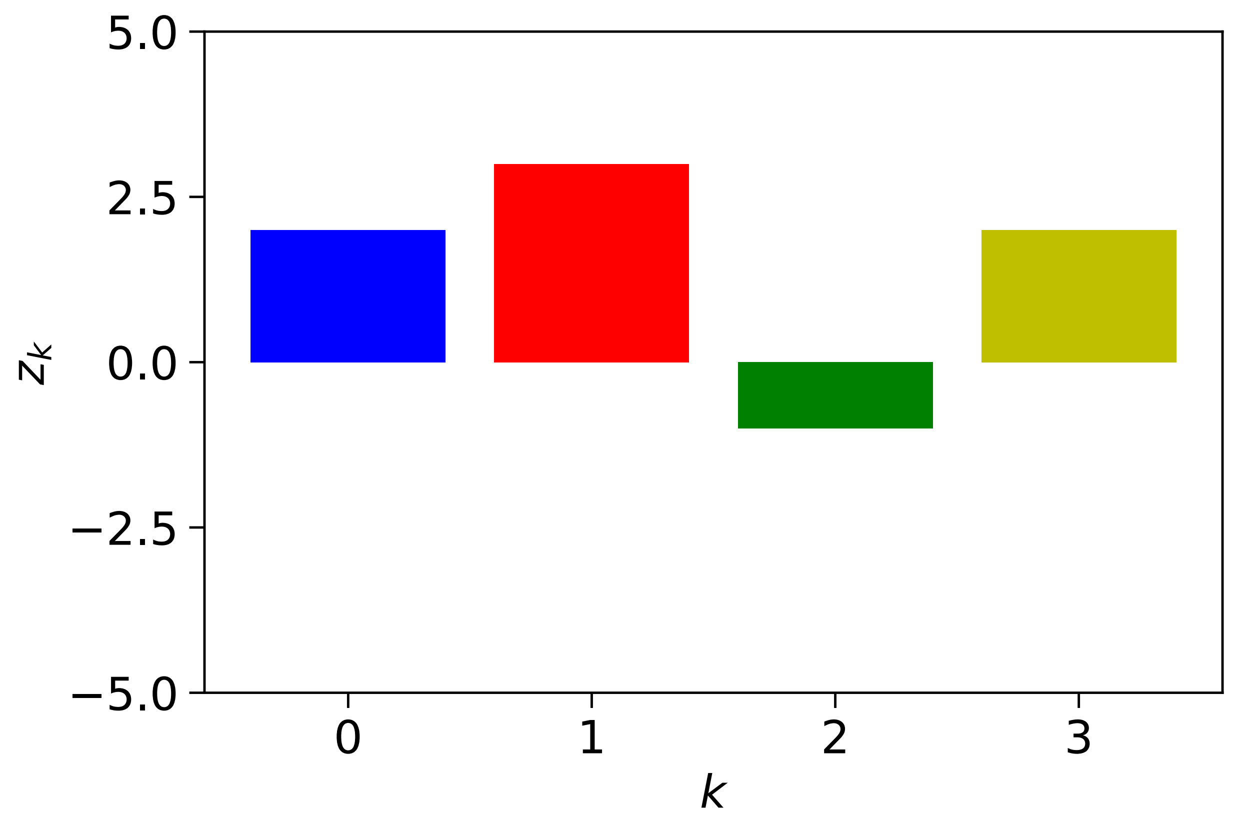
- Logistic regression naturally extends to multi-class problems
- In the binary setting, we just had $f(\mathbf{x}) \in \mathbb{R}$ as the log-odds for class 1
- We now have $f(\mathbf{x}) \in \mathbb{R}^K$. There are the **logits** for each class
- They are unnormalised log-probabilities; a generalisation of log-odds
- Let's store the actual probabilities in a vector $\mathbf{p}$ and relate these to the logits via some function S

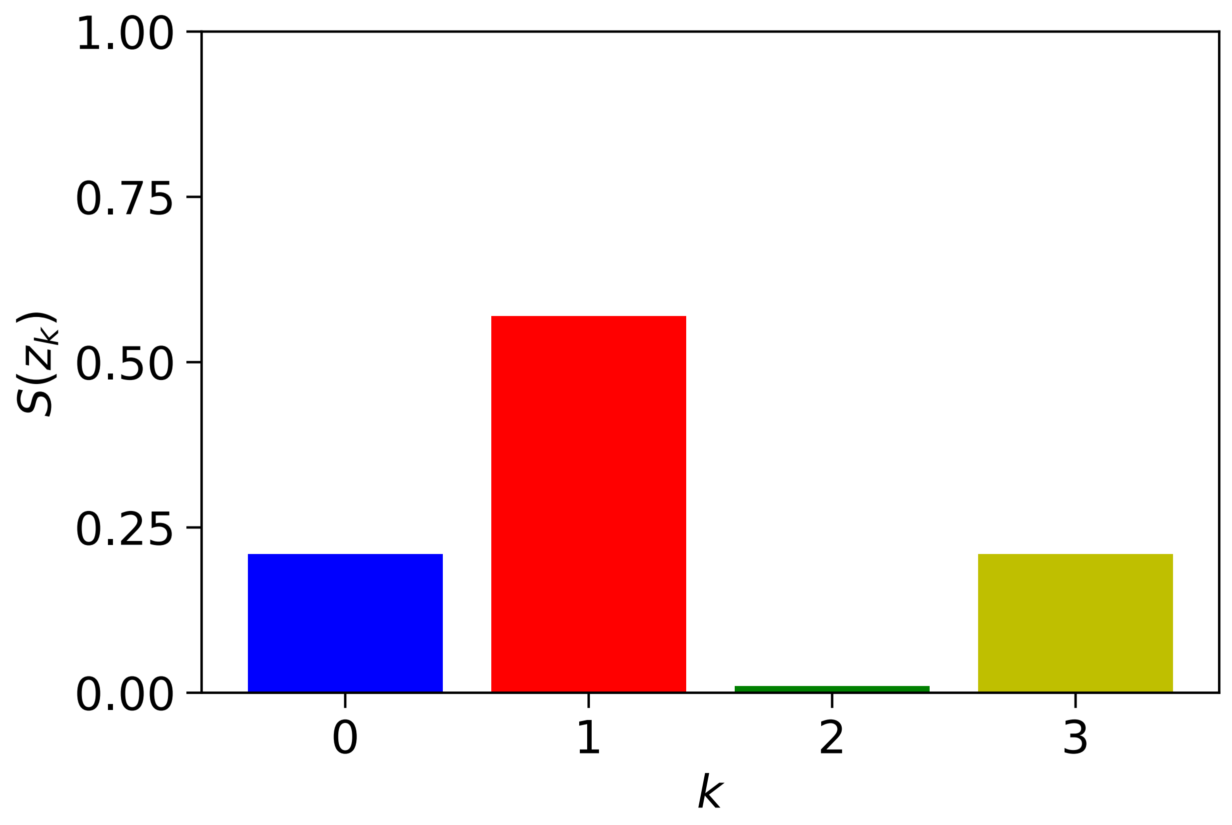
$$\mathbf{p} = \begin{bmatrix} p(y = 0 \,|\, \mathbf{x}) \\ p(y = 1 \,|\, \mathbf{x}) \\ p(y = 2 \,|\, \mathbf{x}) \\ \vdots \\ p(y = K - 1 \,|\, \mathbf{x}) \end{bmatrix} = S(f(\mathbf{x}))$$

Softmax

- $\mathbf{p}$ must sum to 1 so we need a function that normalises $f(\mathbf{x})$
- We will use the softmax function S which squashes $f(\mathbf{x})$ so it sums to 1 and all values are between 0 and 1

$$S(\mathbf{z}) = S(\begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{K-1} \end{bmatrix}) = \begin{bmatrix} \frac{\exp z_0}{\sum_{k=0}^{K-1} \exp z_k} \\ \frac{\exp z_1}{\sum_{k=0}^{K-1} \exp z_k} \\ \vdots \\ \frac{\exp z_{K-1}}{\sum_{k=0}^{K-1} \exp z_k} \end{bmatrix}$$





Learning for multinominal logistic regression

- We have $\{(\mathbf{x}^{(n)}, y^{(n)})\}_{n=1}^{N}$ and $f(\mathbf{x}) = \mathbf{W}\mathbf{x} + \mathbf{b}$ and want to solve $\underset{\mathbf{W},\mathbf{b}}{\text{minimise}}\ L_{log}$

- If we **one-hot encode** our labels as vectors then we can write the log loss as
$$L_{log} = \frac{1}{N}\sum^{N} - \mathbf{y}^{(n)\top} \log \mathbf{p}^{(n)}$$

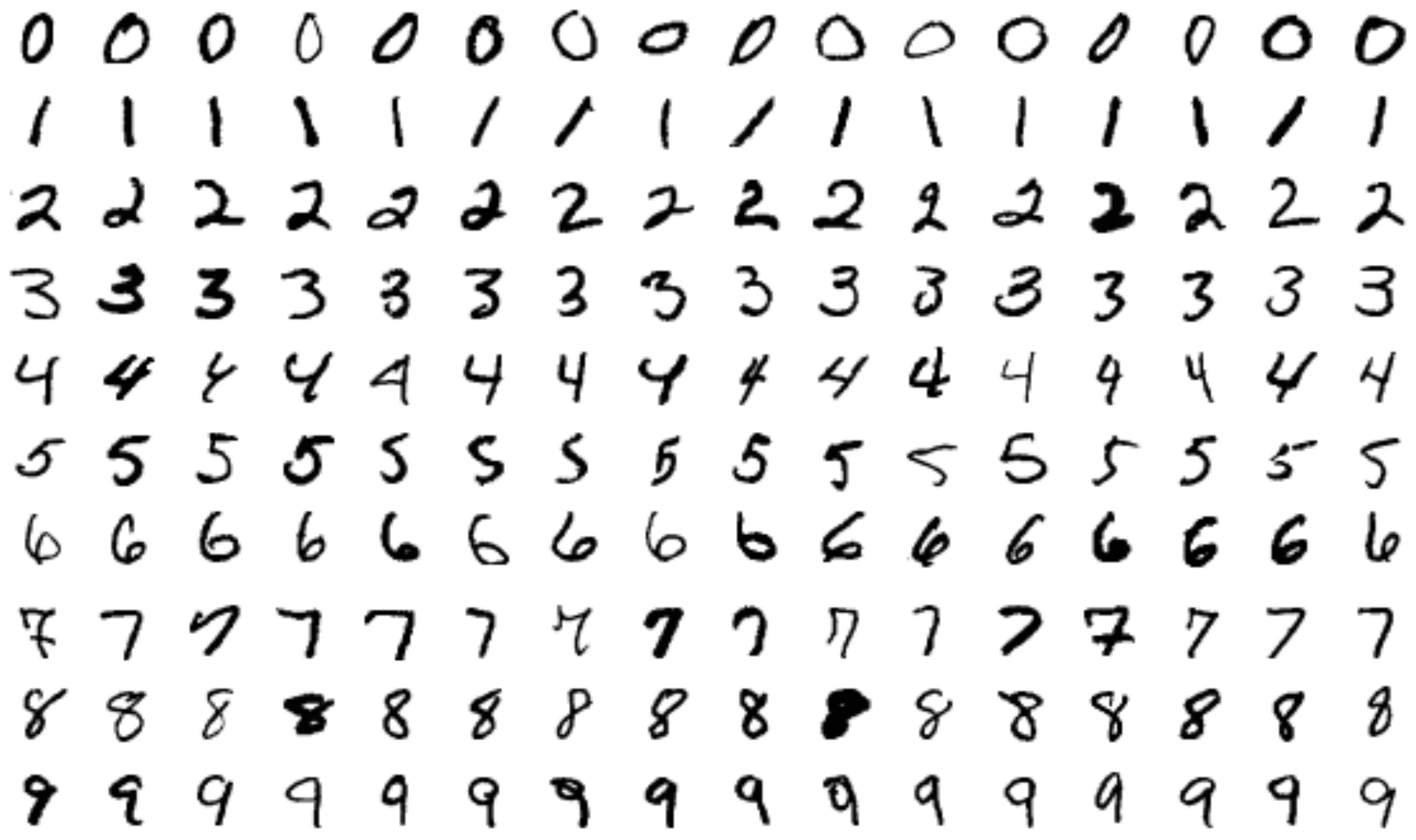
- $\mathbf{y} \in \mathbb{R}^K$ is a one-hot encoding of y which is 1 for the element corresponding to class k and zero elsewhere e.g. for $K = 6$ and $y^{(t)} = 2$ we have $\mathbf{y}^{(t)} = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]^\top$

- We can use a gradient-based optimiser with $\nabla_{\mathbf{W}} L_{log}$ and $\nabla_{\mathbf{b}} L_{log}$

$$\nabla_{\mathbf{W}} L_{log} = \frac{1}{N}\Big[\sum_n (\mathbf{p}^{(n)} - \mathbf{y}^{(n)})\mathbf{x}_n^\top\Big] \qquad \nabla_{\mathbf{b}} L_{log} = \frac{1}{N}\Big[\sum_n (\mathbf{p}^{(n)} - \mathbf{y}^{(n)})\Big]$$

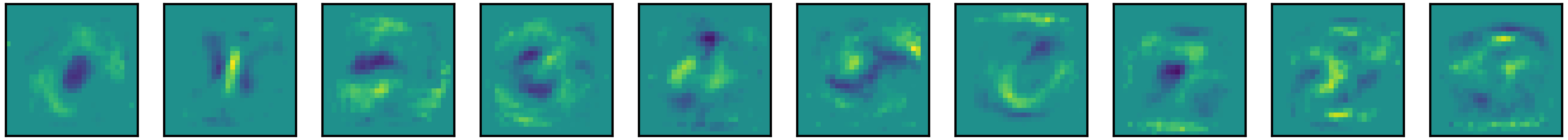
See Murphy p346 for a derivation, noting differences in notation.

Digit classification on MNIST



https://en.wikipedia.org/wiki/MNIST_database#/media/File:MnistExamples.png

- MNIST dataset has 60k images (50k train, 10k test)
- Images are 28×28 so can vectorise to get $\mathbf{x} \in \mathbb{R}^{784}$
- Each image is labelled as a digit 0-9 so $y \in \mathbb{Z}^{+}_{<10}$
- Let's perform multinomial logistic regression with L1 regularisation
- Predict according to most probable class: $\hat{y} = \underset{k}{\operatorname{argmax}}\, \mathbf{p} = \underset{k}{\operatorname{argmax}}\, f(\mathbf{x})$
- Test accuracy: 89.4% (or test error: 10.6%)



0 1 2 3 4 5 6 7 8 9

These are the columns of $\mathbf{W}$ displayed as images

Inspired by https://scikit-learn.org/stable/auto_examples/linear_model/plot_sparse_logistic_regression_mnist.html

Summary

- We have seen that a linear classifier is a linear model plus a threshold function and looks at its decision boundary
- We have found out how to perform logistic regression for binary classification
- We have briefly looked at perceptrons and SVMs
- We have seen how to adapt logistic regression for multi-class classification